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JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and ornament thereunto."—BACON.

VOL. XIV.

LONDON:
CHARLES & EDWIN LAYTON, 150, FLEET STREET.
EDINBURGH: JOHN GRANT. NEW YORK: H. BAILLIÈRE, 290, BROADWAY.
PARIS: J. B. BAILLIÈRE, LIBRAIRE, RUE HAUTEFEUILLE.
HAMBURG: PERTHES, BESSER, & MAUKE.

1869.

LONDON:
PRINTED BY CHARLES AND EDWIN LAYTON,
FLEET STREET AND GOUGH SQUARE.

CONTENTS OF VOL. XIV.

JOURNAL

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INSTITUTE OF ACTUARIES

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*On Interpolation; an Essay containing a simple Exposition of the Theory in its most useful practical applications, together with a general and complete Demonstration of the Methods of Quintisection of Briggs and Mouton for equal intervals; and of the process explained by Newton in his "Principia," for intervals of any magnitude whatever.** *By* M. FRÉD. MAURICE.

Translated from the French by THOMAS B. SPRAGUE, M.A., *and* J. HILL WILLIAMS, *Vice-Presidents of the Institute of Actuaries.*

THE calculation of extensive tables of logarithms could not fail to give rise to the method of interpolation; and such indeed was its origin. After Napier had made his memorable discovery, and he and Briggs had recognized the advantages that would result from the adoption of 10 as the base of a system of logarithms, it was Briggs who courageously undertook the immense labour required for calculating those famous tables, published in London in 1624, which have not been surpassed in extent and accuracy by any subsequent work. It is well known, however, that Briggs left a great *hiatus* in those tables, and that the logarithms of numbers between 20,000 and 90,000 are not to be found in them; but scarcely were the first tables printed, when Briggs undertook with fresh energy the computation of the Logarithms of the Trigonometrical Lines,

* From the *Connaissance des Temps* for the year 1847.

and he was on the point of terminating this vast enterprise when arrested by death. It was his friend Gellibrand who completed the work, and gave it to the world in 1633.

The great service rendered to science by this distinguished computer is not confined to the exact determination to a great number of decimal places of the logarithms calculated by him, and of their differences of several orders. Briggs has besides shown himself to be a mathematician of great skill, by his learned prefaces to the two great works to which we have referred. In these he has developed a host of ingenious methods invented by himself, with the view of combining rigorous exactness with facility of computation; and, with that object, he has given various remarkable methods which enabled him to verify his results to any degree of accuracy required.

One of the most remarkable of these is the singular process he explains (chapter xiii. of the Preface to his first tables, and chapter xii. of the Preface to his Trigonometrical Tables) for interpolating four intermediate values between each adjacent two of a series obtained by direct calculation, both the original and the interpolated values being separated by equal intervals. To this explanation he adds that, in publishing to 14 decimal places the logarithms of the 30,000 numbers comprised between 1 and 20,000, and between 90,000 and 100,000, it was his intention to give, by this process, an easy means of determining the 70,000 logarithms remaining to be calculated in order to complete those of the first hundred thousand natural numbers; and it may well be thought that he had already employed it to compute some of the logarithms contained in his valuable tables—we mean the Tables of Logarithms of Numbers, edited by himself in 1624.

This method, which is very remarkable, and preceded by nearly half a century the researches of Mouton and Regnauld upon a similar method of interpolation, is given by the author without demonstration. He only lays down the highly complicated rules of his method, and that in a manner wholly wanting in the clearness which, since the time of Euler, analysis has learnt how to throw on most subjects of which it treats. Furthermore, the process itself is more singular than easy of application: there seems nothing in it to guide the computer in choosing the limit of the corrections to which he must confine himself; and, moreover, the differences, of various orders, of the function under consideration, having reference to different values, might give rise to some confusion in practice.

To this defect in the explanation we must, no doubt, attribute the silence of mathematicians as to an invention so remarkable and of such merit. No one drew attention to its originality until, after the lapse of two centuries, Legendre made it the subject of a learned paper in the additions to the *Connaissance des Temps* for 1817. He there demonstrated, by means of a rigorous modern analysis, the reasons of the rules given by Briggs for deducing the differences to be employed in interpolation from those which he calls *mean differences*; showing that these last have, in fact, necessary relations with the differences of the original terms.

In that paper, the shortness of which is a matter of regret, Legendre confesses that his own demonstration is wanting in simplicity, and expresses a hope that some other proof may be found more closely akin to that which the author must undoubtedly have discovered, although he did not publish it. In the present paper we purpose to answer that appeal; and we believe we have succeeded, without using any process or method of investigation not known to mathematicians living in 1620, contemporary with Kepler and Harriot, and before Descartes and Fermat.

With this view, it appeared advisable, before giving our elementary demonstration of Briggs's curious process, to examine all the methods of investigation which were within his reach when about half way through his great work; and this will be the object of our first section. In the second, we shall describe, in Briggs's own words, that process which Legendre has deemed so worthy of attention; and in the third, it will be completely demonstrated, by a purely arithmetical method, whose only resemblance to modern analysis is due to the convenient notation so generally used at the present time.

We have contrasted with this rather obscure method the clear and convenient one published in 1670 by Mouton. He was a priest of Lyons, and the author of an astronomical work on the diameters of the sun and moon, containing several ideas which, considering the period in which he wrote, are very remarkable. But this method, which Mouton states was generalized by his friend and countryman Regnauld, is only described by him at some length, and is not demonstrated. Lalande was the first who undertook (in 1761) to demonstrate it for the three first orders of differences; and no further progress had been made, when Lagrange, in the Berlin *Memoirs* for 1792–3, published a demonstration as general as it was learned. Shortly after, Prony took this excellent method for the basis of the immense work executed under his direction, for

computing the great logarithmic tables called *Tables Du Cadastre* (Government survey), and, we are told, developed it in the preface in all its details. Those tables, however, which surpass both in extent and accuracy all that have been published, only exist in two manuscript copies, carefully preserved in different places, so that Prony's analysis has never been given to the public. It is true that Lacroix, in the third volume of his great treatise, has briefly given an outline of the method and of its demonstration; but as it is far from elementary, we trust that our readers will not consider as useless the more simple and complete exposition of it which we give in the fourth section. We shall there further show that Briggs's and Mouton's methods, notwithstanding their great dissimilarity, lead to exactly the same results, not only in their numerical applications, but likewise in the general formulæ. As, moreover, we have not omitted to give such details as may be practically useful in interpolating, it will be seen that the clear and easy process of Mouton offers unusual advantages from the symmetry of all its operations.

Hitherto we have only considered the case when the intervals between the given values are equal; but other cases occur in practice. For example, in some astronomical observations it happens that the times between the successive observations are unequal; and then, taking the time as the variable, the values given by observation are not equidistant, and the problem is to find from the given data the value of the observed element at any given time.

Newton gave the first solution of this problem in a lemma in his *Principia*, which, according to his custom, he did not stop to demonstrate; and Laplace, who adopted that solution in his method of calculating the orbit of a comet, also confined himself to its simple enunciation. We devote our fifth and last section to the general consideration of cases of this kind; and after giving a solution based on the same purely algebraical reasoning we have used throughout, we shall conclude with an easy demonstration of Newton's solution, under the form given to it by Laplace in his *Mécanique Céleste.*

Lastly, in an Appendix, we have considered more particularly the formulæ most frequently used in astronomical interpolations; and we have deduced them also from the same simple principles. We would particularly draw attention to the manner in which we there treat a formula which Stirling, in his excellent work on interpolation, had only obtained by induction, and to the additions we have given to M. Bessel's demonstration of that useful formula. The combined simplicity and clearness of all these calculations will, it is hoped, secure for them a careful perusal.

§ I.

1. Some persons who have carefully examined the prefaces of Briggs, have seen therein reason for thinking that he had anticipated Pascal, Wallis and Newton, in the discovery of the celebrated Binomial Theorem. Dr. Charles Hutton, for instance, in the introduction to his Tables, states this explicitly, and endeavours to explain how the fact may be reconciled with Newton's acknowledged claim to that most valuable discovery. We shall see, however, that Briggs may have arrived at a very analogous formula by a widely different course of reasoning, which would not necessarily require him to consider the raising of a binomial to a given power, either integral or fractional. We shall do our author full justice by allowing that he was led by the very nature of his investigations to find general formulæ showing the relations between the equidistant terms of any given series and their differences of any order. It is evident that the logarithms of successive numbers are quantities of this nature.

2. Thus, if we have a series of given quantities

$$u_0,\ u_1,\ u_2,\ u_3,\ \ldots\ldots,\ u_x,\ \ldots\ldots,$$

corresponding to values of the variable, x,

$$0,\ 1,\ 2,\ 3,\ \ldots\ldots\ x,\ \ldots\ldots,$$

it is easily seen that we may assume the general term of the series to be given by the formula

$$u_x = u_0 + \mathrm{A}x + \mathrm{B}.x(x-1) + \mathrm{C}.x(x-1)(x-2) + \mathrm{D}.x(x-1)(x-2)(x-3) + \ldots$$

where the coefficients A, B, C, D, ... are to be determined by means of the conditions (obtained by putting x equal to 1, 2, 3, ... successively)

$$u_1 = u_0 + \mathrm{A}, \qquad u_3 = u_0 + 3\mathrm{A} + 6\mathrm{B} + 6\mathrm{C},$$
$$u_2 = u_0 + 2\mathrm{A} + 2\mathrm{B}, \qquad u_4 = u_0 + 4\mathrm{A} + 12\mathrm{B} + 24\mathrm{C} + 24\mathrm{D}, \ldots.,$$

Hence we deduce, representing by the symbol δ the difference of two of the quantities, and by δ^n, generally, a difference of the nth order,

$$\mathrm{A} = u_1 - u_0 = \delta u_0;$$

$$\mathrm{B} = \frac{u_2 - 2u_1 + u_0}{2} = \frac{u_2 - u_1 - (u_1 - u_0)}{2} = \frac{\delta(u_1 - u_0)}{2} = \frac{\delta^2 u_0}{1.2};$$

$$\mathrm{C} = \frac{u_3 - 3u_2 + 3u_1 - u_0}{1.2.3} = \frac{\delta^3 u_0}{1.2.3};$$

$$\mathrm{D} = \frac{u_4 - 4u_3 + 6u_2 - 4u_1 + u_0}{1.2.3.4} = \frac{\delta^4 u_0}{1.2.3.4}.$$

Substituting these values, the formula for the general term becomes

$$(E)\qquad u_x=u_0+\frac{x}{1}\cdot\delta u_0+\frac{x(x-1)}{1.2}\cdot\delta^2u_0+\frac{x(x-1)(x-2)}{1.2.3}\cdot\delta^3u_0$$
$$+\frac{x(x-1)(x-2)(x-3)}{1.2.3.4}\cdot\delta^4u_0+\ldots$$

3. This expression calls for two remarks:—

The first is, that the series may come to an end, either because x is a finite integer, and then all the coefficients after $\frac{x(x-1)\ldots(x-\overline{x-1})}{1.2\ldots x}$ will of necessity vanish; or because we meet with constant differences of the nth order, in which case all the terms after $\frac{x(x-1)\ldots(x-n+1)}{1.2\ldots n}\cdot\delta^n u_0$ will disappear, because $\delta^{n+1}u_0=0$, &c.

The second remark is, that we have not proved that the form which we have seen to prevail in the case of a few terms, will be generally true whatever the number of terms. But a very simple consideration will satisfy us on this point. Suppose that x is increased by unity; we shall have

$$u_{x+1}=u_x+\delta u_x;$$

$$\therefore\quad u_{x+1}=u_0+\frac{x}{1}\cdot\delta u_0+\frac{x(x-1)}{1.2}\cdot\delta^2u_0+\frac{x(x-1)(x-2)}{1.2.3}\cdot\delta^3u_0+\ldots$$
$$+1.\delta u_0+\frac{x}{1}\cdot\delta^2u_0\qquad+\frac{x(x-1)}{1.2}\cdot\delta^3u_0\qquad+\ldots$$
$$=u_0+\frac{x+1}{1}\cdot\delta u_0+\frac{(x+1)x}{1.2}\cdot\delta^2u_0+\frac{(x+1)x(x-1)}{1.2.3}\cdot\delta^3u_0+\ldots$$

Thus u_{x+1} is expressed by a series in perfect conformity with that of the formula (E); from which it may be strictly concluded that if the formula is proved for any term, u_x, it is true also for the next term, u_{x+1}; and by successive inductions it is true for all positive and integral values of x.

This formula (E) is so related to that of the binomial theorem, that a century and a half after Briggs's time it was at last written in the abridged form

$$u_x=\{1+\delta\}^x u_0.$$

In fact, the formula for u_x results rigorously from the product of u_0 by the expansion of the xth power of the binomial $(1+\delta)$.

4. We are able to see also, when the coefficients A, B, C, D, ... have been determined, that the successive differences of u are expressed in series containing the successive values of u itself, with alternate signs and coefficients similar to those of the series

(E); so that, for example, up to the nth order inclusive, we have obtained the expression

$$\text{(D)} \quad \delta^n u_0 = u_n - \frac{n}{1} \cdot u_{n-1} + \frac{n(n-1)}{1.2} \cdot u_{n-2} - \frac{n(n-1)(n-2)}{1.2.3} \cdot u_{n-3} + \ldots$$

A line of reasoning similar to that of the preceding article will prove that this general form is also true when n becomes $n+1$.

In fact, $\delta^{n+1}u_0 = \delta^n(u_1 - u_0)$. But the value of $\delta^n u_1$ will be deduced from that of $\delta^n u_0$ just found, by writing u_{n+1} for u_n, u_n for u_{n-1}, &c. We shall then have

$$\begin{aligned} \delta^{n+1}u_0 = u_{n+1} - \frac{n}{1} \cdot u_n + \frac{n(n-1)}{1.2} \cdot u_{n-1} - \frac{n(n-1)(n-2)}{1.2.3} \cdot u_{n-2} + \ldots . \\ - 1.u_n + \frac{n}{1} \cdot u_{n-1} \qquad - \frac{n(n-1)}{1.2} \cdot u_{n-2} \qquad + \ldots . \\ = u_{n+1} - \frac{n+1}{1} u_n + \frac{(n+1)n}{1.2} \cdot u_{n-1} - \frac{(n+1)n(n-1)}{1.2.3} \cdot u_{n-2} + \ldots \end{aligned}$$

But this expression being of exactly the same form as (D), we see that if that formula is true for the nth difference, it is also true for the $(n+1)$th difference; and by successive inductions it is true for any order of differences whatever.

This new formula received, about the same time as the previous one, the following concise and convenient form—

$$\delta^n u = (u-1)^n,$$

where care must be taken, in the expansion of the second member, to write throughout u_n instead of u^n, u_{n-1} instead of u^{n-1}, &c.; and in the last term, instead of 1 or u^0, to write u_0.

5. We have considered, (art. 2) a series of given quantities, $u_0, u_1, u_2, \ldots, u_x, \ldots$ or values of the function u_x, corresponding to the values of the variable $0, 1, 2, \ldots, x, \ldots$; and we have obtained the formula (E) for the general term u_x.

Let us now consider the series $u_0, u_h, u_{2h}, \ldots, u_z, \ldots$ where h is an integer, and where we must have $z = xh$, x being also an integer.

Then it is evident that the formula for the general term u_z will be deduced from that for u_x or from (E) by substituting for x in the second member its value $\frac{z}{h}$; for x takes the values $0, 1, 2, \ldots x, \ldots$ when z is according to our hypothesis made equal to $0, h, 2h, \ldots xh, \ldots$ successively; only it will be necessary to put Δu_0 or $u_h - u_0$, and its successive differences $\Delta^2 u_0$, $\Delta^3 u_0, \ldots$ in the place of δu_0, or $u_1 - u_0$, and its successive differences $\delta^2 u_0$, $\delta^3 u_0, \ldots$

Making these changes, the formula becomes

$$\text{(F)}\quad u_z = u_0 + \frac{z}{h}\Delta u_0 + \frac{z(z-h)}{1.2.h^2}\Delta^2 u_0 + \frac{z(z-h)(z-2h)}{1.2.3h^3}\Delta^3 u_0 + \ldots.$$

which may also be obtained directly, in just the same manner as the formula (E) was first obtained (art. 2), if we represent by Δu_0 the difference $u_h - u_0$.

6. This formula (F), which for values of z equal to 0, h, $2h$. . . gives

$$\text{(Z)}\quad \begin{cases} u_0 = u_0 \\ u_h = u_0 + \Delta u_0 \\ u_{2h} = u_0 + 2\Delta u_0 + \Delta^2 u_0 \\ \&\text{c.} = \quad \&\text{c.} \end{cases}$$

will of necessity coincide with that deduced from the formula (E), by making x equal to 0, h, $2h$, . . . successively. Conversely, (E) must also coincide with (F), when z is made equal to 0, 1, 2, . . . successively, which gives

$$\text{(Z)}'\quad \begin{cases} u_0 = u_0 \\ u_1 = u_0 + \frac{1}{h}\Delta u_0 + \frac{1(1-h)}{1.2.h^2}\Delta^2 u_0 + \frac{1(1-h)(1-2h)}{1.2.3.h^3}\Delta^3 u_0 + \ldots. \\ u_2 = u_0 + \frac{2}{h}\Delta u_0 + \frac{2(2-h)}{1.2.h^2}\Delta^2 u_0 + \frac{2(2-h)(2-2h)}{1.2.3.h^3}\Delta^3 u_0 + \ldots. \\ \&\text{c.} = \quad \&\text{c.} \end{cases}$$

We thus conclude that the use of the single formula (F) will give the values of u_z for all integral values of z. Consequently, if we have computed a series of values of a function u_z, such as are shown in (Z), we see from (Z)′ that we can determine, by means of the known quantities, u_0, Δu_0, $\Delta^2 u_0$ . . . . $\Delta^{x-1} u_0$, the $h-1$ values of u_z comprised between u_0 and u_h, or between u_h and u_{2h}, &c., and this is called *interpolating*; *i.e.*, if we have a series of values of u_z corresponding to values of z, that differ by the constant quantity h, we know how to insert, between any two adjacent values, $(h-1)$ intermediate values, equidistant like the first.

7. Similarly if the original successive values of z had been 0, 1, 2, 3, . . ., instead of 0, h, $2h$, . . . we should have had a formula exactly similar to (E), and which would be written thus:—

$$\text{(E}'\text{)}\quad u_z = u_0 + \frac{z}{1}\delta u_0 + \frac{z(z-1)}{1.2}\delta^2 u_0 + \frac{z(z-1)(z-2)}{1.2.3}\delta^3 u_0 + \ldots.$$

and the known values of u_z would be

$$\begin{aligned} u_0 &= u_0 \\ u_1 &= u_0 + \delta u_0 \\ u_2 &= u_0 + 2\delta u_0 + \delta^2 u_0 \\ \&\text{c.} &= \quad \&\text{c.} \end{aligned}$$

But if we now suppose z to have the values $0, \frac{1}{h}, \frac{2}{h}, \ldots \frac{x}{h}, \ldots$ successively, it will be evidently sufficient to put, for z, in (E′) its general value $\frac{x}{h}$; for (E′) is really the same formula as (E): they are both dependent upon increments of the independent variable equal to unity; so that in the form which (E′) then takes, increments of x equal to unity will give the successive values of u_z.

We shall have then

$$(\mathrm{F}') \quad u_z = u_{\frac{x}{h}} = u_0 + \frac{x}{h}\,\delta u_0 + \frac{x(x-h)}{1.2.h^2}\,\delta^2 u_0 + \frac{x(x-h)(x-2h)}{1.2.3.h^3}\,\delta^3 u_0 + \ldots.$$

a formula, which, making x equal to $0, h, 2h, \ldots$, will give again the preceding values of $u_0, u_1, u_2, \ldots$, namely,

$$u_0,\; u_0 + \delta u_0,\; u_0 + 2\delta u_0 + \delta^2 u_0, \ldots.$$

Consequently, if we have a table showing, like the formula (E) the values of u_x for values of x equal to $0, 1, 2, 3, \ldots$, so that we know the values of $u_0, \delta u_0, \delta^2 u_0, \delta^3 u_0, \ldots$, the formula (F′) will give the $(h-1)$ values $u_{\frac{1}{h}}, u_{\frac{2}{h}}, \ldots u_{\frac{h-1}{h}}$, intermediate between u_0 and $u_{\frac{h}{h}}$ or u_1, or enable us to *interpolate* between u_0 and u_1 these $(h-1)$ values which are, like the original values, equidistant. It is evident that we may do the same for the second interval comprised between $u_{\frac{h}{h}}$ or u_1, and $u_{\frac{2h}{h}}$ or u_2; for the third, and for all subsequent intervals.

8. Such are the means of calculation which we may suppose Briggs to have possessed when he began to work at his great tables. In these operations we find really nothing beyond the general arithmetic then known; as far as theory goes, they only introduce the simple and precise ideas of the differences of various orders of given quantities, and the reciprocal relations between such quantities and differences.

We shall, however, soon see that these formulæ and ideas were more than sufficient to lead him to the singular method of interpolation which we are about to explain, and which we shall first present as it appears in the 13th chapter of the Preface to the *Arithmetica Logarithmica*, published at London in 1624.

9. We have said that these means were *more than sufficient* to lead Briggs to his method. We shall see in fact that it is highly probable that he never obtained the formulæ (F) and (F′); at all events, they are not required in order to obtain his results. More-

over, it may be remarked, that if he had only had before him the second of the formulæ (Z′) in art. 6, which gives

$$\delta u_0 = \frac{1}{h}\Delta u_0 + \frac{1(1-h)}{1.2.h^2}\Delta^2 u_0 + \dots$$

he would have been led to think that $\delta^n u_0$ must depend upon $\Delta^n u_0$, $\Delta^{n+1} u_0, \dots$; and that, consequently, if $\Delta^n u_0$ were a *constant* difference, it was unnecessary to notice $\delta^{n+1} u_0$, $\delta^{n+2} u_0, \dots$. But, as will be seen in articles 11, 18 and 28, Briggs carries the formula for $\delta^4 u_0$ up to the *ten thousandth part* of $\delta^{20} u_0, \dots$, whilst, supposing $\Delta^5 u_0$ constant, he should only have taken $\delta^5 u_0$ into consideration, and should have neglected all differences of a higher order.

But these observations will arise more naturally as we proceed. Let us now hear Briggs's own description of his curious process.

§ II.

10. In the 13th chapter of the Preface to his Tables, Briggs wished to give the means of finding the logarithms which he had not calculated, namely, those of the numbers between 20,000 and 90,000; and for that purpose he proposed the following problem:— "Given a series of equidistant numbers and their logarithms, to find the logarithms of the four numbers interpolated at equal intervals between each adjacent two of the given numbers."

Thus, for example, Briggs having calculated the logarithms of all the numbers below 20,000, easily obtained, by adding the logarithm of 5 to those of the numbers 4,001, 4,002, 4,003, . . . , the logarithms of the numbers 20,005, 20,010, 20,015, . . . ; but he had still to find a means of obtaining, without too much labour, the logarithms of the four whole numbers comprised between each two of the last numbers.

But let the author speak for himself. "Take," he goes on to say, "the first, second, third, fourth, &c., differences of the logarithms of 2,115, 2,120, 2,125, . . . , and divide the first differences by 5, the second by 5^2, the third by 5^3, . . . (or, which comes to the same thing, multiply them respectively by ·2, by ·04, by ·008, &c.), and these quotients (or products) will be what I call the *mean differences* of the corresponding orders." Briggs then gives examples of the manner of obtaining them; and afterwards proceeds to *correct* them, so as to be able to employ them in his investigations.

11. Having found that the fifth differences of the logarithms of the numbers 2,115, 2,120, 2,125, . . . , are constant, Briggs observes that the *mean* differences of the fourth and fifth orders cannot be *corrected;* for, he says, the sixth and seventh differences are *nil*; adding, "*omnis autem differentiarum correctio fit per subductionem differentiarum alternarum magis remotarum et correctarum*" (every correction of the differences is made by subtracting the alternate corrected differences of higher orders) (Preface, p. 27, near the bottom). This passage appears at first sight to involve a vicious circle, inasmuch as it requires, for the calculation of the corrections, that we should have differences already corrected. But this difficulty disappears when we notice that Briggs begins by obtaining differences which, requiring no correction, may be said to be already corrected; so that when he subsequently makes all these corrections, commencing with the highest order, he obtains successively the corrected differences he requires, whose order in effect *alternates* by increasing at each step by two units (See Articles 15 to 18 and 28 below, and compare them).

Briggs continues:—"Thus the subtraction of the seventh differences corrects the fifth differences, that of the sixth differences corrects the fourth, and so on." Therefore, in the case under consideration, we may take the *mean* fourth and fifth differences for the *corrected* fourth and fifth differences.

As to the mean third differences, we are to correct them by the subtraction of *three* corrected fifth differences, and Briggs gives a numerical example.

The mean second differences are corrected by the subtraction of *two* corrected fourth differences, and also of $\frac{7}{5}$ of the sixth difference, if it have any sensible value. Briggs here gives a fresh example.

Lastly, we are to correct the mean first differences by deducting therefrom *one* corrected third difference, and $\frac{1}{5}$ of the fifth difference; and Briggs gives another calculation as an example.

"Such," adds Briggs, "is the course to be pursued in correcting the differences, whatever may be their order, commencing always with the highest order." He thereupon gives a table showing all the subtractions required to correct the successive mean differences of every order from the first to the twentieth. It shows, for example, that the difference of the fourth order is corrected by the subtraction of the following multiples of the higher corrected dif-

ferences, each difference being represented by the exponent of its order placed within brackets:

$$\left[\begin{array}{c} 4(6)+6{\cdot}8(8)+6{\cdot}4(10)+3{\cdot}64(12)+1{\cdot}28(14) \\ +{\cdot}272(16)+{\cdot}032(18)+{\cdot}0016(20) \end{array}\right].$$

12. Briggs then gives some practical suggestions, intended to prevent confusion in the application of his rules; and, as an example, he finds the logarithms of the eight numbers comprised between 2,115 and 2,120 and between 2,120 and 2,125. For this purpose he forms a table, in which are arranged in different places both the given logarithms and their differences of the first four orders, corrected as previously explained; those of the fifth order being neglected as insignificant. He finally states that the required logarithms will be found by *adding* the fourth differences to the third, the sum of these to the second differences, the new sum to the first differences, and the sum thus found to the preceding logarithm.

But this table appears complicated and badly arranged, if we may venture to say so. For we must *subtract,* and not add the differences, since the differences of the logarithms of increasing numbers decrease in value; and Briggs himself does this. It is, moreover, impossible not to observe the needless fulness of a process by which, the *mean* differences of the fourth order being reduced to two figures in the thirteenth and fourteenth place of decimals, we are told to correct them by means of differences smaller in themselves, which, however, have been previously computed to the sixth, eighth, ... twentieth, place of decimals.

It is not our object here to praise this process, which, as we have just seen, Briggs greatly modifies in his actual computations, but only to explain and prove it. For this is all which the author has given by way of explanation, and that is but little. He only adds that this method, which may be called *quintisection,* applies, with the requisite alterations, to the steps necessary for effecting *trisection* and *septisection,* but that he greatly prefers the method of *quintisection.*

§ III.

13. If Briggs had been in possession of the formula (F), he might have thought of using it to accomplish the object he had in view in his method of *quintisection.* In fact, making $h=5$, and and treating $\Delta^5 u$ as a constant quantity, the quotients $\frac{\Delta^n u_0}{h^n}$, in that

formula would be his *mean differences.* For brevity, write Δ^n for $\frac{\Delta^n u_0}{h^n}$; then applying (F) to find the values of $u_1, u_2, u_3, \ldots,$ as far as terms involving Δ^5, we might find $\delta u_0, \delta u_1, \delta u_2, \ldots, \delta^2 u_0, \delta^2 u_1, \ldots,$ in terms of $\Delta^1, \Delta^2, \ldots, \Delta^5$, whence, conversely, we could deduce the values of $\Delta^1, \Delta^2, \ldots, \Delta^5$, in terms of $\delta u_0, \delta^2 u_0, \ldots$ But there is no reason to believe that Briggs was acquainted with that formula; and moreover, in following the course just pointed out, he would have encountered long and complicated calculations wholly devoid of symmetry.

14. The formula (D) of art. 4, which is much more simple, was more likely to have suggested itself to him, as offering great advantages for the object he had in view. In fact, when the variable receives successive increments of 5, that formula gives

$$(A)\quad \begin{cases} \Delta u_0 = u_5 - u_0 \\ \Delta^2 u_0 = u_{10} - u_5 - (u_5 - u_0) = P, \text{ suppose} \\ \Delta^3 u_0 = u_{15} - u_{10} - (u_{10} - u_5) - P = Q - P \\ \Delta^4 u_0 = u_{20} - u_{15} - (u_{15} - u_{10}) - 2Q + P = R - Q - (Q - P) \\ \Delta^5 u_0 = u_{25} - u_{20} - (u_{20} - u_{15}) - 3R + 3Q - P \\ \qquad = S - R - 2(R - Q) + (Q - P) \end{cases}$$

On the other hand, the formula (E) gives, putting $x=5$,

$$u_5 - u_0 = 5(\delta u_0 + 2\delta^2 u_0 + 2\delta^3 u_0 + \delta^4 u_0 + \frac{1}{5}\delta^5 u_0),$$

which may be reduced to

$$(b)\qquad u_5 - u_0 = 5(\delta u_2 + \delta^3 u_1 + \frac{1}{5}\delta^5 u_0).$$

But we have, by the first principles of the method of differences,

$$\begin{aligned} u_5 - u_0 &= \delta u_0 + \delta u_1 + \delta u_2 + \delta u_3 + \delta u_4 \\ u_{10} - u_5 &= \delta u_5 + \delta u_6 + \delta u_7 + \delta u_8 + \delta u_9 \\ u_{15} - u_{10} &= \delta u_{10} + \delta u_{11} + \delta u_{12} + \delta u_{13} + \delta u_{14} \\ \&c. &= \&c. \end{aligned}$$

Here it is at once seen that each of the preceding equations may be deduced from the preceding one by increasing the variable by 5; and we must therefore get expressions for $u_{10} - u_5$, $u_{15} - u_{10}, \ldots$ precisely similar to the value given by (b) for $u_5 - u_0$. Thus,

$$\begin{aligned} u_{10} - u_5 &= 5(\delta u_7 + \delta^3 u_6 + \frac{1}{5}\delta^5 u_5) \\ u_{15} - u_{10} &= 5(\delta u_{12} + \delta^3 u_{11} + \frac{1}{5}\delta^5 u_{10}) \\ \&c. &= \&c. \end{aligned}$$

But we can readily satisfy ourselves that we shall have likewise

$$u_7 - u_2 = 5(\delta u_4 + \delta^3 u_3 + \frac{1}{5}\delta^5 u_2)$$

$$u_{13} - u_8 = 5(\delta u_{10} + \delta^3 u_9 + \frac{1}{5}\delta^5 u_8)$$

$$\&c. \quad = \quad \&c.$$

and we shall be justified in concluding generally that

$$\text{(B)} \qquad u_m - u_{m-5} = 5(\delta u_{m-3} + \delta^3 u_{m-4} + \frac{1}{5}\delta^5 u_{m-5})$$

from which we can easily deduce

$$\text{(C)} \qquad \delta^n(u_m - u_{m-5}) = 5(\delta^{n+1} u_{m-3} + \delta^{n+3} u_{m-4} + \frac{1}{5}\delta^{n+5} u_{m-5}).$$

15. These preliminaries being settled, let us return to the relation (b). It will give, dividing by 5,

$$\frac{u_5 - u_0}{5} = \frac{\Delta u_0}{5} = \Delta^1 = \delta u_2 + \delta^3 u_1 + \frac{1}{5}\delta^5 u_0;$$

from which we obtain

$$\delta u_2 = \Delta^1 - \delta^3 u_1 - \frac{1}{5}\delta^5 u_0.$$

But this is the first expression in Briggs's table, which has been verified by Legendre.

(16.) The second of the formulæ (A) and the relation (B) give in turn

$$(x) \qquad \Delta^2 u_0 = 5\left\{\delta(u_7 - u_2) + \delta^3(u_6 - u_1) + \frac{1}{5}\delta^5(u_5 - u_0)\right\}.$$

But by means of the relation (C), we get

$$\delta(u_7 - u_2) = 5\left(\delta^2 u_4 + \delta^4 u_3 + \frac{1}{5}\delta^6 u_2\right)$$

$$\delta^3(u_6 - u_4) = 5\left(\qquad \delta^4 u_3 + \delta^6 u_2 + \frac{1}{5}\delta^8 u_1\right)$$

$$\frac{1}{5}\delta^5(u_5 - u_0) = 5\left(\qquad \frac{1}{5}\delta^6 u_2 + \frac{1}{5}\delta^8 u_1 + \frac{1}{5^2}\delta^{10} u_0\right)$$

Adding and multiplying by 5,

$$\Delta^2 u_0 = 5^2(\delta^2 u_4 + 2\delta^4 u_3 + 1{\cdot}4\delta^6 u_2 + {\cdot}4\delta^8 u_1 + {\cdot}04\delta^{10} u_0);$$

or, dividing by 5^2 and transposing,

$$(2) \qquad \delta^2 u_4 = \Delta^2 - 2\delta^4 u_3 - 1{\cdot}4\delta^6 u_2 - {\cdot}4\delta^8 u_1 - {\cdot}04\delta^{10} u_0.$$

This is the second formula in Briggs's table, which has also been verified by Legendre.

17. The third of the formulæ (A) and the relation (B) will give similarly

$$\Delta^3 u_0 = 5\left\{\delta(u_{12}-u_7)+\delta^3(u_{11}-u_6)+\frac{1}{5}\delta^5(u_{10}-u_5)\right\}-\Delta^2 u_0;$$

or, by virtue of (C) and substituting for $\Delta^2 u_0$ its value as given by (x),

$$(y)\quad \Delta^3 u_0 = 5^2\left\{\delta^2(u_9-u_4)+\delta^4(u_8-u_3)+\frac{1}{5}\delta^6(u_7-u_2)\right\}$$

$$+5^2\left\{\delta^4(u_8-u_3)+\delta^6(u_7-u_2)+\frac{1}{5}\delta^8(u_6-u_1)\right\}$$

$$+5^2\left\{\frac{1}{5}\delta^6(u_7-u_2)+\frac{1}{5}\delta^8(u_6-u_1)+\frac{1}{5^2}\delta^{10}(u_5-u_0)\right\}$$

$$=5^2\left\{\delta^2(u_9-u_4)+2\delta^4(u_8-u_3)+\frac{7}{5}\delta^6(u_7-u_2)\right.$$

$$\left.+\frac{2}{5}\delta^8(u_6-u_1)+\frac{1}{5^2}\delta^{10}(u_5-u_0)\right\}$$

$$=5^3\left[\delta^3 u_6+\delta^5 u_5+\frac{1}{5}\delta^7 u_4+2\left(\delta^5 u_5+\delta^7 u_4+\frac{1}{5}\delta^9 u_3\right)\right.$$

$$+\frac{7}{5}\left(\delta^7 u_4+\delta^9 u_3+\frac{1}{5}\delta^{11}u_2\right)+\frac{2}{5}\left(\delta^9 u_3+\delta^{11}u_2+\frac{1}{5}\delta^{13}u_1\right)$$

$$\left.+\frac{1}{5^2}\left(\delta^{11}u_2+\delta^{13}u_1+\frac{1}{5}\delta^{15}u_0\right)\right]$$

whence, dividing by 5^2, reducing and transposing, we get

$$(3)\quad \delta^3 u_6 = \Delta^3 - 3\delta^5 u_5 - 3{\cdot}6\delta^7 u_4 - 2{\cdot}2\delta^9 u_3 - {\cdot}72\delta^{11}u_2 - {\cdot}12\delta^{13}u_1 - {\cdot}008\delta^{15}u_0,$$

and we thus have the third formula in Briggs's table, the last which has been verified and demonstrated by Legendre.

18. We pass now to the fourth of the formulæ (A). It gives

$$\Delta^4 u_0 = \{u_{20}-u_{15}-(u_{15}-u_{10})\}-\Delta^3 u_0-(\Delta^3 u_0-\Delta^2 u_0);$$

whence, by means of (B) and (C), substituting the values of $\Delta^2 u_0$ and $\Delta^3 u_0$, as found from (x) and (y),

$$\Delta^4 u_0 = 5\left\{\delta(u_{17}-u_{12})+\delta^3(u_{16}-u_{11})+\frac{1}{5}\delta^5(u_{15}-u_{10})\right.$$

$$\left.-\delta(u_{12}-u_7)-\delta^3(u_{11}-u_6)-\frac{1}{5}\delta^5(u_{10}-u_5)\right\}$$

$$-5\left\{\delta(u_{12}-u_7)+\delta^3(u_{11}-u_6)+\frac{1}{5}\delta^5(u_{10}-u_5)\right.$$

$$\left.-\delta(u_7-u_2)-\delta^3(u_6-u_1)-\frac{1}{5}\delta^5(u_5-u_0)\right\}$$

$$
\begin{aligned}
&=5^2\Big\{\delta^2(u_{14}-u_9)+\delta^4(u_{13}-u_8)+\frac{1}{5}\,\delta^5(u_{12}-u_7)\\
&\qquad\qquad -\delta^2(u_9-u_4)-\delta^4(u_8-u_3)-\frac{1}{5}\,\delta^6(u_7-u_2)\Big\}\\
&\quad+5^2\Big\{\delta^4u_{13}-u_8)+\delta^6(u_{12}-u_7)+\frac{1}{5}\,\delta^8(u_{11}-u_6)\\
&\qquad\qquad -\delta^4(u_8-u_3)-\delta^6(u_7-u_2)-\frac{1}{5}\,\delta^8(u_6-u_1)\Big\}\\
&\quad+5^2\Big\{\frac{1}{5}\,\delta^6(u_{12}-u_7)+\frac{1}{5}\,\delta^8(u_{11}-u_6)+\frac{1}{5^2}\,\delta^{10}(u_{10}-u_5)\\
&\qquad\qquad -\frac{1}{5}\,\delta^6(u_7-u_2)-\frac{1}{5}\,\delta^8(u_6-u_1)-\frac{1}{5^2}\,\delta^{10}(u_5-u_0)\Big\}\\
&=5^3\Big[\delta^3u_{11}+\delta^5u_{10}+\frac{1}{5}\,\delta^7u_9+\delta^5u_{10}+\delta^7u_9+\frac{1}{5}\,\delta^9u_8\\
&\quad+\frac{1}{5}\Big(\delta^7u_9+\delta^9u_8+\frac{1}{5}\,\delta^{11}u_7\Big)-\Big(\delta^3u_6+\delta^5u_5+\frac{1}{5}\,\delta^7u_4\Big)\\
&\quad-\Big(\delta^5u_5+\delta^7u_4+\frac{1}{5}\,\delta^9u_3\Big)-\frac{1}{5}\Big(\delta^7u_4+\delta^9u_3+\frac{1}{5}\,\delta^{11}u_2\Big)\\
&\quad+\delta^5u_{10}+\delta^7u_9+\frac{1}{5}\,\delta^9u_8+\delta^7u_9+\delta^9u_8+\frac{1}{5}\,\delta^{11}u_7\\
&\quad+\frac{1}{5}\Big(\delta^9u_8+\delta^{11}u_7+\frac{1}{5}\,\delta^{13}u_6\Big)-\Big(\delta^5u_5+\delta^7u_4+\frac{1}{5}\,\delta^9u_3\Big)\\
&\quad-\Big(\delta^7u_4+\delta^9u_3+\frac{1}{5}\,\delta^{11}u_2\Big)-\frac{1}{5}\Big(\delta^9u_3+\delta^{11}u_2+\frac{1}{5}\,\delta^{13}u_1\Big)\\
&\quad+\frac{1}{5}\Big(\delta^7u_9+\delta^9u_8+\frac{1}{5}\,\delta^{11}u_7\Big)+\frac{1}{5}\Big(\delta^9u_8+\delta^{11}u_7+\frac{1}{5}\,\delta^{13}u_6\Big)\\
&\quad+\frac{1}{5^2}\Big(\delta^{11}u_7+\delta^{13}u_6+\frac{1}{5}\,\delta^{15}u_5\Big)-\frac{1}{5}\Big(\delta^7u_4+\delta^9u_3+\frac{1}{5}\,\delta^{11}u_2\Big)\\
&\quad-\frac{1}{5}\Big(\delta^9u_3+\delta^{11}u_2+\frac{1}{5}\,\delta^{13}u_1\Big)-\frac{1}{5^2}\Big(\delta^{11}u_2+\delta^{13}u_1+\frac{1}{5}\,\delta^{15}u_0\Big)\Big]\\
&=5^3\Big\{\delta^3(u_{11}-u_6)+3\delta^5(u_{10}-u_5)+\frac{18}{5}\,\delta^7(u_9-u_4)+\frac{11}{5}\,\delta^9(u_8-u_3)\\
&\qquad\qquad+\frac{18}{5^2}\,\delta^{11}(u_7-u_2)+\frac{3}{5^2}\,\delta^{13}(u_6-u_1)+\frac{1}{5^3}\,\delta^{15}(u_5-u_0)\Big\}\\
&=5^4\big\{\delta^4u_8+4\delta^6u_7+6{\cdot}8\delta^8u_6+6{\cdot}4\delta^{10}u_5+3{\cdot}64\delta^{12}u_4+1{\cdot}28\delta^{14}u_3\\
&\qquad\qquad+{\cdot}272\delta^{16}u_2+{\cdot}332\delta^{18}u_1+{\cdot}0016\delta^{20}u_0\big\}
\end{aligned}
$$

whence, dividing by 5^4 and transposing,

$$
\begin{aligned}
(4)\qquad \delta^4u_8=\Delta^4&-4\delta^6u_7-6{\cdot}8\delta^8u_6-6{\cdot}4\delta^{10}u_5-3{\cdot}64\delta^{12}u_4\\
&-1{\cdot}28\delta^{14}u_3-{\cdot}272\delta^{16}u_2-{\cdot}032\delta^{18}u_1-{\cdot}0016\delta^{20}u_0;
\end{aligned}
$$

and this is in effect the fourth formula in Briggs's table, which we have inserted at art. 11, from page 29 of his Preface.*

19. We shall not proceed any further with these calculations, which present no difficulty. Their law is evident, and they are eminently symmetrical. It must be admitted they are simple, although far from brief. All the operations are purely arithmetical, and depend on the proper use of the relations (B) and (C), applied to two direct consequences of the formulæ (D) and (E). The minute resolutions required return with such symmetry that we can almost write down the results mechanically.

We therefore think it highly probable that this was the course followed by Briggs. It does not presuppose the use of any knowledge superior to that of a contemporary of Kepler and Harriot, whose works appeared about fifteen or twenty years before the more advanced era of Descartes and Fermat.

But, as we have previously stated (arts. 9 and 12), the process appears to us rather curious than useful. There appears to be nothing to guide the computer as to the limit he should prescribe to himself in making his corrections; and, moreover, the differences of various orders having reference to different terms in the original series might give considerable trouble in practice.†

20. On the contrary, nothing can be more lucid and convenient than the method published in 1670 by Mouton; and although, in the last century, two different demonstrations of it have been given, we trust that the more simple and complete exposition we are about to give in the following section may not be considered superfluous.

§ IV.

21. "Given a series of quantities such that the differences of any given order are constant, to find any number whatever of intermediate values that shall follow the same law."

Mouton was the first person who regarded interpolation from this point of view. The process published by him, after Regnauld, is a general one, but very long. He solves the problem by showing how to find by means of the differences of the original quantities, those of the interpolated terms; these terms being then found from the differences by successive additions. Nothing can be simpler than this process.

* M. Maurice's demonstration fails to exhibit the general law of this formula for $\delta^4 u_8$. This defect we hope to supply in our next number.—Ed. *J. I. A.*

† It appears to us that M. Maurice does not do full justice to Briggs's method in consequence of his not observing that the differences $\delta^4 u_8$, $\delta^6 u_7$, $\delta^8 u_6$, $\delta^{10} u_5$ .... are all "central differences" opposite the same value, u_{10}.—Ed. *J. I. A.*

The only difficulty then is to find these differences of the interpolated terms, the number $(h-1)$ of the terms interpolated and the order (n) of the constant differences in the given series, being any whatever.

To solve it, Mouton and Regnauld had arrived at the following principle, probably suggested to them by the examination of several particular cases:—*When the values of a function corresponding to equidistant values of the variable have their differences of any given order constant, then the values interpolated between them at equal intervals will also have their differences of the same order constant.*

This principle is true, but only in the case of rational and integral functions, as in Mouton's problem, and as we will now demonstrate.

22. Let u be a function of x, such that

$$u=\alpha x^m+\beta x^p+\gamma x^q+\ldots.$$

where the indices are all positive integers, and $m>p>q>\ldots$

Let us consider the relation $u=\alpha x^m$. Since m is the highest power of x in the value of u, the consideration of the other terms is useless for our purpose. Supposing $\delta x=h$, we shall have

$$\Delta u=a\{(x+h)^m-x^m\}=amhx^{m-1}+\mathrm{A}x^{m-2}+\mathrm{B}x^{m-3}+\ldots.$$

from which we deduce

$$\Delta^2 u=amh\Delta x^{m-1}+\mathrm{A}\Delta x^{m-2}+\ldots.$$

and consequently

$$\Delta^2 u=amh(m-1)hx^{m-2}+\mathrm{A}'x^{m-2}+\ldots.$$

In the same manner we shall find

$$\Delta^3 u=amh(m-1)h(m-2)hx^{m-3}+\mathrm{A}''x^{m-4}+\ldots.$$

$$\vdots$$

$$\Delta^n u=am(m-1)(m-2)\ldots(m-n+1)h^n x^{m-n}+\mathrm{A}^{(n-1)}x^{m-n-1}+\ldots$$

Now, if $m=n$, since $n>p>q>\ldots$, we see that if we had considered the complete value of u, all the terms of $\Delta^n u$ after the first would necessarily have disappeared; so that we should have had simply

$$\Delta^n u=n(n-1)(n-2)\ldots.2ah^n$$

for the value of the *constant* difference; for $\Delta^{n+1}u=0$.

It is also evident that if, taking $\delta x=1$, we represent by δu the difference $(x+1)^m-x^m$, we should find in just the same way

$$\delta^n u=n(n-1)(n-2)\ldots.2.1.a,$$

and we should consequently have

$$\delta^n u = \frac{\Delta^n u}{h^n}.$$

Thus the supposed principle is demonstrated, and the value of $\delta^n u$ is found.

23. Assuming then that we have a series of quantities, u_0, u_h, u_{2h}, . . . , which give the differences Δu_0, $\Delta^2 u_0$, $\Delta^3 u_0$, . . . $\Delta^n u_0$, of which the last is constant, we see from what precedes that we should know also $\delta^n u_0$, which relates to the series u_0, u_1, u_2, . . . u_h, . . . and is, like $\Delta^n u_0$, constant. Having also u_0, if we could find δu_0, $\delta^2 u_0$, . . . $\delta^{n-1} u_0$, we might by simple additions obtain the quantities u_1, u_2, u_3, . . . , u_{h-1}, . . .

To establish this, let us confine our attention, like Briggs and Mouton, to the case when $n=h=5$, and bear in mind that in general

$$(m) \qquad u_{m+1} = u_m + \delta u_m.$$

Thus, when u represents a number, it will be seen that each value is formed by the addition of the preceding one and of its difference; and as we know that $u_m = (1+\delta)^m u_0$, it is clear that we shall have

$$\delta u_m = u_{m+1} - u_m = \{(1+\delta) - 1\}\{(1+\delta)^m u_0\} = \delta(1+\delta)^m u_0.$$

Consequently, δu_m will be expressed in terms of δu_0, and its successive differences; so that we shall have, m being any whole number,

$$\delta u_m = u_{m+1} - u_m = \delta u_0 + m\delta^2 u_0 + \frac{m(m-1)}{1.2}\delta^3 u_0 + \frac{m(m-1)(m-2)}{1.2.3}\delta^4 u_0 + \ldots$$

It will therefore be sufficient, as already stated, to know how to express the differences δu_0, $\delta^2 u_0$, $\delta^3 u_0$, $\delta^4 u_0$, in terms of the differences Δ ($\delta^5 u_0$ being a constant and its value known), in order to be able to calculate all the interpolated numbers.

Assume for a moment that we know these four quantities, δu_0, . . . $\delta^4 u_0$, and we shall see how easily we may obtain the interpolated numbers, by virtue of an obvious consequence of the relation (m) which gives

$$(n) \qquad \delta^k u_{m+1} = \delta^k u_m + \delta^{k+1} u_m.$$

24. Let us consider the following table, the law of which is evident :—

$$\begin{array}{llllll}
u_0, & & & & & \\
u_1 & \delta u_0, & & & & \\
u_2 & \delta u_1 & \delta^2 u_0, & & & \\
u_3 & \delta u_2 & \delta^2 u_1 & \delta^3 u_0, & & \\
u_4 & \delta u_3 & \delta^2 u_2 & \delta^3 u_1 & \delta^4 u_0, & \\
u_5 & \delta u_4 & \delta^2 u_3 & \delta^3 u_2 & \delta^4 u_1 & \delta^5 u_0 \ (\doteq \text{constant, by hypothesis}) \\
u_6 & \delta u_5 & \delta^2 u_4 & \delta^3 u_3 & \delta^4 u_2 & \delta^5 u_0, \\
u_7 & \delta u_6 & \delta^2 u_5 & \delta^3 u_4 & \delta^4 u_3 & \delta^5 u_0, \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Every term in this table will be known, if we know only those at the end of each horizontal line. For, by virtue of the relation (n), in accordance with which it has been formed, each term being equal to the sum of the two which stand respectively above it and on the right of it, we see that

$$\begin{aligned}
u_1 &= u_0 + \delta u_0 \\
\delta u_1 &= \delta u_0 + \delta^2 u_0 \\
u_2 &= u_1 + \delta u_1 \\
\delta^2 u_1 &= \delta^2 u_0 + \delta^3 u_0 \\
\delta u_2 &= \delta u_1 + \delta^2 u_1 \\
u_3 &= u_2 + \delta u_2 \\
\delta^3 u_1 &= \delta^3 u_0 + \delta^4 u_0 \\
\delta^2 u_2 &= \delta^2 u_1 + \delta^3 u_1 \\
\delta u_3 &= \delta u_2 + \delta^2 u_2 \\
u_4 &= u_3 + \delta u_3
\end{aligned}$$

and so on *ad infinitum*. It is furthermore easy to see how we must proceed if the constant differences are of either a higher or a lower order than $\delta^5 u_0$.

Thus, each of the interpolated values will be formed by adding to the previous one its difference, which itself will be obtained by simple, regular and successive additions, and will involve δu_0, $\delta^2 u_0$, $\delta^3 u_0$ . . . up to the order for which these differences are constant.

25. We have lastly to find δu_0, $\delta^2 u_0, \ldots \delta^n u_0$, in terms of Δu_0, $\Delta^2 u_0, \ldots, \Delta^n u_0$, supposing that $\Delta^n u_0$ is constant.

Let there be two series of values of the function, in one of which every value is known, but in the other only every hth value is known, namely:—

$$\begin{array}{lllll}
v_0, & v_1, & v_2, & v_3, \ldots\ldots v_x, & \\
u_0,\ u_1,\ u_2, \ldots\ldots & u_h,\ u_{h+1}, \ldots & u_{2h},\ u_{2h+1}, \ldots\ldots & u_{3h}, \ldots\ldots u_{xh}, \ldots\ldots &
\end{array}$$

It is clear that if $v_0 = u_0$, $v_1 = u_h$, we shall have generally, $v_x = u_{xh}$.

Denoting by δ the differences of u, and by Δ the differences of v, we have

$$u_1=u_0+\delta u_0, \qquad v_1=v_0+\Delta v_0=u_0+\Delta v_0,$$

whence $$\Delta v_0=v_1-u_0=u_h-u_0.$$

We therefore have, since $u_h=(1+\delta)^h u_0$,

$$\Delta v_0=\{(1+\delta)^h-1\}u_0.$$

Let us confine ourselves, for simplicity, to the case of $h=5$. We shall see however that the course of procedure is quite general. Let us also put $(1+\delta)^5=\beta$. We shall then easily get

$$\begin{array}{lll}
v_0=u_0 & v_1-v_0=\Delta v_0=(\beta-1)u_0=\Delta u_0 & \Delta v_1-\Delta v_0=\Delta^2 v_0=(\beta-1)^2 u_0=\Delta^2 u_0 \\
v_1=u_h\;=\beta u_0 & \Delta v_1=\beta(\beta-1)u_0 & \Delta^2 v_1=(\beta-1)^3 u_0 \\
v_2=u_{2h}=\beta^2 u_0 & \Delta v_2=\beta^2(\beta-1)u_0 & \Delta^2 v_2=(\beta-1)^4 u_0 \\
v_3=u_{3h}=\beta^3 u_0 & \Delta v_3=\beta^3(\beta-1)u_0 & \Delta^2 v_3=(\beta-2)^5 u_0 \\
v_4=u_{4h}=\beta^4 u_0 & \Delta v_4=\beta^4(\beta-1)u_0 & \ldots\ldots \\
v_5=u_{5h}=\beta^5 u_0 & \ldots\ldots & \\
\ldots\ldots & &
\end{array}$$

$$\begin{array}{lll}
v_1-\Delta^2 v_0=\Delta^3 v_0=(\beta-1)^3 u_0=\Delta^2 u_0 & \Delta^3 v_1-\Delta^3 v_0=\Delta^4 v_0=(\beta-1)^4 u_0=\Delta^4 u_0 & \Delta^4 v_1-\Delta^4 v_0=\Delta^5 v_0=(\beta-1)^5 u_0=\Delta^5 u_0 \\
\Delta^3 v_1=\beta(\beta-1)^3 u_0 & \Delta^4 v_1=\beta(\beta-1)^4 u_0 & \ldots\ldots\ldots\ldots \\
\Delta^3 v_2=\beta^2(\beta-1)^3 u_0 & \ldots\ldots & \\
\ldots\ldots & &
\end{array}$$

We have thus shown that

$$\begin{aligned}
\Delta v_0 &=\Delta u_0 =(\beta-1)u_0 =\{1+\delta)^5-1\}u_0 \\
\Delta^2 v_0&=\Delta^2 u_0=(\beta-1)^2 u_0=\{1+\delta)^5-1\}^2 u_0 \\
\Delta^3 v_0&=\Delta^3 u_0=(\beta-1)^3 u_0=\{1+\delta)^5-1\}^3 u_0 \\
\Delta^4 v_0&=\Delta^4 u_0=(\beta-1)^4 u_0=\{1+\delta)^5-1\}^4 u_0 \\
\Delta^5 v_0&=\Delta^5 u_0=(\beta-1)^5 u_0=\{1+\delta)^5-1\}^5 u_0
\end{aligned}$$

and it is clear that, whatever integers m and h may be, we shall have

$$\Delta^m u_0=\{(1+\delta)^h-1\}^m u_0;$$

whence expanding $\{(1+\delta)^h-1\}$, we have

$$\text{(P)}\quad \left[h\delta+\frac{h(h-1)}{1.2}\delta^2+\frac{h(h-1)(h-2)}{1.2.3}\delta^3+\ldots\right]^m$$
$$=h^m\delta^m+\mathrm{A}\delta^{m+1}+\mathrm{B}\delta^{m+2}+\mathrm{C}\delta^{m+3}+\ldots$$

If then we can by any means determine the numerical values of the coefficients A, B, C, . . . ; we shall only have to multiply the second member of (P) by u_0, and we shall have $\Delta^m u_0$ in terms of $\delta^m u_0$, $\delta^{m+1}u_0$, . . . As to the determination of those coefficients, we shall see that the law of their formation is very simple; and we confine ourselves here to remarking that if the differences δ^m be constant, the differences of higher orders will vanish; so that we again get $\Delta^m u_0=h^m\delta^m u_0$.

26. But the problem is not to find the differences Δ in terms of the differences δ; it is the converse problem we have to solve.

For that purpose, since $v_1=(1+\Delta)v_0=u_h=(1+\delta)^h u_0$, we get

$$(1+\delta)u_0=(1+\Delta)^{\frac{1}{h}}v_0;$$

or, since $u_0=v_0$,

$$\delta u_0=\{(1+\Delta)^{\frac{1}{h}}-1\}u_0.$$

Reasoning as in art. 25, we shall have

$$\delta^m u_0=\{(1+\Delta)^{\frac{1}{h}}-1\}^m u_0.$$

Hence, when $m=1$, we have

$$\delta u_0=\{1+\Delta)^{\frac{1}{h}}-1\}u_0=\left\{\frac{\Delta}{h}+\frac{1(1-h)}{1.2}\cdot\frac{\Delta^2}{h^2}+\frac{1(1-h)(1-2h)}{1.2.3}\cdot\frac{\Delta^3}{h^3}+\dots\right\}u_0.$$

But, if we take $h=5$, suppose $\Delta^5 u_0$ constant, and for brevity put k for $\frac{\Delta}{5}$, we can very easily calculate the numerical values of the coefficients of $k^2 \dots k^5$, and we thus get

$$(1) \qquad \delta u_0=(k+ak^2+bk^3+ck^4+dk^5)u_0.$$

But $\delta^2 u_0$ will depend on the sqnare of the value of δ thus found; and since we need not consider higher powers of k than k^5, putting $k+ak^2=\alpha$, $bk^3+ck^4+dk^5=\beta$, we see at once that in the expansion of $\delta^2 u_0=(\alpha+\beta)^2 u_0$, we may confine our attention to the following

$$\delta^2 u_0=(\alpha^2+2\alpha\beta)u_0,$$

which gives

$$(2) \qquad \delta^2 u_0=\{k^2+2ak^3+(a^2+2b)k^4+2(ab+c)k^5\}u_0.$$

In the same way

$$\delta^3 u_0=(\alpha+\beta)^3 u_0$$

reduces to

$$\delta^3 u_0=(\alpha^3+3\alpha^2\beta)u_0,$$

whence we obtain

$$(3) \qquad \delta^3 u_0=\{k^3+3ak^4+3(a^2+b)k^5\}u_0.$$

In the same way, $\delta^4 u_0=\alpha^4 u_0$ leads to

$$(4) \qquad \delta^4 u_0=(k^4+4ak^5)u_0,$$

and lastly

$$(5) \qquad \delta^5 u_0=k^5 u_0=\frac{\Delta^5 u_0}{5^5},$$

as we have already more than once seen.

27. The actual calculation of all these coefficients is as simple as it is rapid. Thus, when $h=5$, we have

$$\frac{1(1-h)}{1.2}=-2=a, \qquad a\frac{1-2h}{3}=+6=b,$$

$$b\frac{1-3h}{4}=-21=c, \qquad c\frac{1-4h}{5}=+79{\cdot}8=d;$$

whence

$$2a=-4, \qquad a^2+2b=+16, \qquad 2(ab+c)=-66,$$

$$3a=-6, \qquad 3(a^2+b)=+30, \qquad 4a=-8.$$

Thus, putting for k its value $\frac{\Delta}{5}$, we shall have

$$(1) \qquad \delta u_0=\frac{\Delta u_0}{5}-2\frac{\Delta^2 u_0}{5^2}+6\frac{\Delta^3 u_0}{5^3}-21\frac{\Delta^4 u_0}{5^4}+79{\cdot}8\frac{\Delta^5 u_0}{5^5}$$

$$(2) \qquad \delta^2 u_0=\frac{\Delta^2 u_0}{5^2}-4\frac{\Delta^3 u_0}{5^3}+16\frac{\Delta^4 u_0}{5^4}-66\frac{\Delta^5 u_0}{5^5}$$

$$(3) \qquad \delta^3 u_0=\frac{\Delta^3 u_0}{5^3}-6\frac{\Delta^4 u_0}{5^4}+30\frac{\Delta^5 u_0}{5^5}$$

$$(4) \qquad \delta^4 u_0=\frac{\Delta^4 u_0}{5^4}-8\frac{\Delta^5 u_0}{5^5}$$

$$(5) \qquad \delta^5 u_0=\frac{\Delta^5 u_0}{5^5}$$

Consequently, since we are supposed to know the numerical values of $\Delta u_0, \ldots \Delta^5 u_0$, we shall very easily find the numerical values of $\delta u_0, \delta^2 u_0, \ldots \delta^5 u_0$; and all that is necessary to form the table in art. 24; so that, the origin u_0 being given, all the interpolated terms will be found by very simple successive additions.

We further notice that while it will never be necessary to consider the differences of a higher order than the seventh or eighth, even in these extreme cases, by following the same process, we shall readily be able to obtain all the numbers necessary for the process of interpolation. As to the value of h, however large it may be, no difficulty can arise.

28. It will be interesting to compare the five formulæ above given with those of Briggs, demonstrated in arts. 15 to 18.

But since Briggs has considered $\delta^m u_n$ we must remember that since

$$u_n=(1+\delta)^n u_0,$$

we have also

$$(\delta) \qquad \delta^m u_n=\delta^m(1+\delta)^n u_0.$$

We must also bear in mind that having proved (art. 25) that

the lowest power of k or $\frac{\Delta}{5}$ in the value of $\delta^m u_0$ is m, it will be useless to carry the expansion of the formula (δ) beyond $\delta^5 u_0$; for $\delta^6 u_0$, $\delta^7 u_0$, &c., will involve no lower order of differences than $\Delta^6 u_0$, $\Delta^7 u_0$, &c., respectively; and these are all zero, because $\Delta^5 u_0$ is constant.

Then, taking the formula (δ), the value (1) of art. 15,

$$\delta u_2 = \Delta^1 - \delta^3 u_1 - \frac{1}{5}\delta^5 u_0$$

will become

$$\delta u_0 + 2\delta^2 u_0 + \delta^3 u_0 = \Delta^1 - \delta^3 u_0 - \delta^4 u_0 - \frac{1}{5}\delta^5 u_0,$$

whence

$$(1') \qquad \delta u_0 = \frac{\Delta u_0}{5} - 2\delta^2 u_0 - 2\delta^3 u_0 - \delta^4 u_0 - \frac{1}{5}\delta^5 u_0.$$

In the same way, the value (2) of art. 16, as far as it need be taken, will be

$$\delta^2 u_4 = \Delta^2 - 2\delta^4 u_3$$

or

$$\delta^2 u_0 + 4\delta^3 u_0 + 6\delta^4 u_0 + 4\delta^5 u_0 = \frac{\Delta^2 u_0}{5^2} - 2\delta^4 u_0 - 6\delta^5 u_0,$$

whence

$$(2') \qquad \delta^2 u_0 = \frac{\Delta^2 u_0}{5^2} - 4\delta^3 u_0 - 8\delta^4 u_0 - 10\delta^5 u_0.$$

The value (3) of art. 17 will similarly give

$$\delta^3 u_6 = \Delta^3 - 3\delta^5 u_5$$

or

$$\delta^3 u_0 + 6\delta^4 u_0 + 15\delta^5 u_0 = \frac{\Delta^3 u_0}{5^3} - 3\delta^5 u_0,$$

whence

$$(3') \qquad \delta^3 u_0 = \frac{\Delta^3 u_0}{5^3} - 6\delta^4 u_0 - 18\delta^5 u_0.$$

The value (4) of art. 18 will give in the same way

$$\delta^4 u_0 = \Delta^4,$$

or

$$\delta^4 u_0 + 8\delta^5 u_0 = \frac{\Delta^4 u_0}{5^4},$$

whence

$$(4') \qquad \delta^4 u_0 = \frac{\Delta^4 u_0}{5^4} - 8\delta^5 u_0.$$

Lastly, Briggs having shown (art. 11) that the last difference needs no correction, it is clear that he would take

$$(5'') \qquad \delta^5 u_0 = \Delta^5 = \frac{\Delta^5 u_0}{5^5}.$$

We shall find then, finally, by successive substitution

$$(4'') \qquad \delta^4 u_0 = \frac{\Delta^4 u_0}{5^4} - 8\frac{\Delta^5 u_0}{5^5}$$

$$(3'') \qquad \delta^3 u_0 = \frac{\Delta^3 u_0}{5^3} - 6\frac{\Delta^4 u_0}{5^4} + 30\frac{\Delta^5 u_0}{5^5}$$

$$(2'') \qquad \delta^2 u_0 = \frac{\Delta^2 u_0}{5^2} - 4\frac{\Delta^3 u_0}{5^3} + 16\frac{\Delta^4 u_0}{5^4} - 66\frac{\Delta^5 u_0}{5^5}$$

$$(1'') \qquad \delta u_0 = \frac{\Delta u_0}{5} - 2\frac{\Delta^2 u_0}{5^2} + 6\frac{\Delta^3 u_0}{5^3} - 21\frac{\Delta^4 u_0}{5^4} + 79{\cdot}8\frac{\Delta^5 u_0}{5^5}$$

and we see that these values (1″) . . . (5″) coincide exactly with the values (1) . . . (5) of art. 27; so that the methods of Briggs and Mouton, notwithstanding their great dissimilarity, lead to identically the same results.

§ V.

29. Hitherto we have supposed the given values to be equidistant; but this is not always the case.

Thus, for example, in observing certain phenomena it sometimes happens that the intervals between the observations are not equal. In that case, the time being taken for the variable, and the results of observation giving the values of the function u_x, those values will correspond to values of the variable, which do not proceed by a constant difference; and the problem consists in finding a general expression for u_x in terms of the given values and one value of x, corresponding to any time we choose.

Let the known values of u_x, corresponding to the values of x, $0, p, q, r, s \ldots$, be $u_0, u_p, u_q, u_r, u_s \ldots$; then we have to find a general expression for u_x, whatever be the value of x.

Reasoning as before, we may assume

$$u_x = u_0 + Bx + Cx(x-p) + Dx(x-p)(x-q) + Ex(x-p)(x-q)(x-r) + \ldots$$

and then denoting by Δ, Δ', Δ'', Δ''' . . . the differences $u_p - u_0$, $u_q - u_0$, $u_r - u_0$, . . . we have

$$u_p - u_0 = \Delta = Bp, \quad u_q - u_0 = \Delta' = Bq + Cq(q-p),$$

$$u_r - u_0 = \Delta'' = Br + Cr(r-p) + Dr(r-p)(r-q) \ldots .$$

Whence we shall find

$$B=\frac{\Delta}{p}$$

$$C=\frac{\Delta'}{q(q-p)}+\frac{\Delta}{p(p-q)}$$

$$D=\frac{\Delta''}{r(r-p)(r-q)}+\frac{\Delta'}{q(q-p)(q-r)}+\frac{\Delta}{p(p-q)(p-r)}$$

$$E=\frac{\Delta'''}{s(s-p)(s-q)(s-r)}+\frac{\Delta''}{r(r-p)(r-q)(r-s)}+\frac{\Delta'}{q(q-p)(q-r)(q-s)}+\frac{\Delta}{p(p-q)(p-r)(p-s)}$$

&c.= &c.

the law of these successive values being simple and obvious. This symmetry, as well as the recurrence of the positive sign which unites all the terms, results from the fact that in the several factors of each denominator we have throughout written that letter first, which is itself one of the factors. Thus, for example, we have written $p(q-p)$ under the form $-p(p-q)$: and similarly with the others.

Having thus obtained the values of the coefficients B, C, D . . . we shall have for the required expression, collecting the terms involving Δ, Δ', Δ'' . . . respectively,

$$(a)\left\{\begin{aligned} u_x=u_0&+\left\{\frac{x}{p}+\frac{x(x-p)}{p(p-q)}+\frac{x(x-p)(x-q)}{p(p-q)(p-r)}+\frac{x(x-p)(x-q)(x-r)}{p(p-q)(p-r)(p-s)}+\dots\right\}\Delta\\ &+\left\{\frac{x(x-p)}{q(q-p)}+\frac{x(x-p)(x-q)}{q(q-p)(q-r)}+\frac{x(x-p)(x-q)(x-r)}{q(q-p)(q-r)(q-s)}+\dots\right\}\Delta'\\ &+\left\{\frac{x(x-p)(x-q)}{r(r-p)(r-q)}+\frac{x(x-p)(x-q)(x-r)}{r(r-p)(r-q)(r-s)}+\dots\right\}\Delta''\\ &+\left\{\frac{x(x-p)(x-q)(x-r)}{s(s-p)(s-q)(s-r)}+\dots\right\}\Delta'''+\dots\end{aligned}\right.$$

30. Let now $X=x(x-p)(x-q)(x-r)(x-s)\dots$, and denote $\frac{X}{x-w}$ by X_w. Then we should have similarly

$$\begin{aligned} P_p&=p(p-q)(p-r)(p-s)\dots\\ Q_q&=q(q-p)(q-r)(q-s)\dots\\ R_r&=r(r-p)(r-q)(r-s)\dots\\ S_s&=s(s-p)(s-q)(s-r)\dots\\ \&c.&=\quad\&c.\end{aligned}$$

If we next consider the coefficients of Δ, Δ', . . . , in the above formula (a) for u_x, we shall notice that the several terms in them can be grouped according to a regular law.

Thus, the first being, for example,

$$\frac{x(x-p)\ldots.(x-t)}{v(v-p)\ldots.(v-t)} \text{ or } V,$$

the sum of the first and second will be

$$V\left(1+\frac{x-v}{v-y}\right)=V\frac{x-y}{v-y}=V_1 \text{ suppose.}$$

The sum of the first, second, and third will be

$$V_1\left(1+\frac{x-v}{v-z}\right)=V_1\frac{x-z}{v-z}=V_2 \text{ suppose,}$$

and so on, where we see the simple factor $x-v$ has disappeared entirely. Treating each of the coefficients of Δ, Δ', Δ'', . . . in this way, we can easily see that the value of u_x can be accurately written in the concise form

$$(a') \qquad u_x=v_0+\frac{X_p}{P_p}\Delta+\frac{X_q}{Q_q}\Delta'+\frac{X_r}{R_r}\Delta''+\frac{X_s}{S_s}\Delta'''+\ldots.$$

In this formula, each of the coefficients of Δ, Δ', Δ'', . . . (which we will denote by a, β, γ, δ, . . . for brevity) is easily calculated by logarithms; and the number of the coefficients will be the same as that of the values p, q, r, . . .

31. This expression for u_x has been readily obtained, because we have considered only the differences u_p-u_0, u_q-u_0, u_r-u_0, &c.; but since these differences of necessity increase, the expression is not convergent, and we shall find a more convenient form by introducing instead of Δ, Δ', Δ'' . . . the differences Δu_0, Δu_p, Δu_q, . . . between the successive values of u_x, which will not necessarily increase.

For this purpose, we notice that

$$\begin{aligned}
\Delta\ &=u_p-u_0=\Delta u_0\\
\Delta' &=u_q-u_0=u_q-u_p+u_p-u_0=\Delta u_p+\Delta u_0\\
\Delta'' &=u_r-v_0=u_r-u_q+u_q-u_p+u_p-u_0=\Delta u_q+\Delta u_p+\Delta u_0
\end{aligned}$$

and so on.

If, then, we make these substitutions in the above value of u_x, denoting as before the coefficients by a, β, γ, . . . , the formula (a') will take the preferable form

$$(a'') \qquad u_x=u_0+(a+\beta+\gamma+\delta+\ldots.)\Delta u_0+(\beta+\gamma+\delta+\ldots.)\Delta u_p$$
$$+(\gamma+\delta+\ldots.)\Delta u_q+(\delta+\ldots.)\Delta u_r+\ldots.$$

32. But we can very easily obtain a symmetrical and more convergent formula. In fact, by the last article

$$
(1)\quad \begin{cases}
\Delta \;\; = \Delta u_0 \\
\Delta' \;\, = \Delta u_p + \Delta u_0 \\
\Delta'' = \Delta u_q + \Delta' \\
\Delta''' = \Delta u_r + \Delta'' \\
\Delta^{\text{IV}} = \Delta u_s + \Delta''' \\
\&\text{c.} = \quad \&\text{c.}
\end{cases}
$$

whence we obtain

$$
(2)\quad \begin{cases}
\Delta u_0 = \Delta, \\
\Delta u_p = \Delta' - \Delta, & \Delta u_p - \Delta u_0 = \Delta' - 2\Delta = \Delta^2 u_0, \\
\Delta u_q = \Delta'' - \Delta', & \Delta u_q - \Delta u_p = \Delta'' - 2\Delta' + \Delta = \Delta^2 u_p, \\
\Delta u_r = \Delta''' - \Delta'', & \Delta u_r - \Delta u_q = \Delta''' - 2\Delta'' + \Delta' = \Delta^2 u_q, \\
\Delta u_s = \Delta^{\text{IV}} - \Delta''', & \Delta u_s - \Delta u_r = \Delta^{\text{IV}} - 2\Delta''' + \Delta'' = \Delta^2 u_r, \\
\cdots\cdots & \cdots\cdots\cdots\cdots\cdots
\end{cases}
$$

$$
(3)\quad \begin{cases}
\Delta^2 u_p - \Delta^2 u_0 = \Delta'' - 3\Delta' + 3\Delta = \Delta^3 u_0, \\
\Delta^2 u_q - \Delta^2 u_p = \Delta''' - 3\Delta'' + 3\Delta' - \Delta = \Delta^3 u_p, \\
\Delta^2 u_r - \Delta^2 u_q = \Delta^{\text{IV}} - 3\Delta''' + 3\Delta'' - \Delta' = \Delta^3 u_q, \\
\cdots\cdots\cdots\cdots\cdots
\end{cases}
$$

$$
(4)\quad \begin{cases}
\Delta^3 u_p - \Delta^3 u_0 = \Delta''' - 4\Delta'' + 6\Delta' - 4\Delta = \Delta^4 u_0, \\
\Delta^3 u_q - \Delta^3 u_p = \Delta^{\text{IV}} - 4\Delta''' + 6\Delta'' - 4\Delta' = \Delta^4 u_p, \\
\cdots\cdots\cdots\cdots\cdots
\end{cases}
$$

$$
(5)\quad \begin{cases}
\Delta^4 u_p - \Delta^4 u_0 = \Delta^{\text{IV}} - 5\Delta''' + 10\Delta'' - 10\Delta' + 5\Delta = \Delta^5 u_0, \\
\cdots\cdots
\end{cases}
$$

From these we readily conclude that

$$
\begin{aligned}
\Delta \;\; &= \Delta u_0, \\
\Delta' \;\, &= \Delta^2 u_0 + 2\Delta u_0, \\
\Delta'' &= \Delta^3 u_0 + 3\Delta^2 u_0 + 3\Delta u_0, \\
\Delta''' &= \Delta^4 u_0 + 4\Delta^3 u_0 + 6\Delta^2 u_0 + 4\Delta u_0, \\
\Delta^{\text{IV}} &= \Delta^5 u_0 + 5\Delta^4 u_0 + 10\Delta^3 u_0 + 10\Delta^2 u_0 + 5\Delta u_0, \\
&\cdots\cdots\cdots\cdots
\end{aligned}
$$

So that, substituting these values in (a'), it will become

$$
\begin{aligned}
(a'')\qquad u_x = u_0 &+ [\alpha + 2\beta + 3\gamma + 4\delta + 5\epsilon + \ldots\ldots].\Delta u_0 \\
&+ [\beta + 3\gamma + 6\delta + 10\epsilon + \ldots\ldots].\Delta^2 u_0 \\
&+ [\gamma + 4\delta + 10\epsilon + \ldots].\Delta^3 u_0 + [\delta + 5\epsilon + \ldots].\Delta^4 u_0 \\
&+ [\epsilon + \ldots\ldots].\Delta^5 u_0 + \ldots\ldots,
\end{aligned}
$$

a perfectly regular series, of which the law is evident, the differences of u_0 in general converging rapidly, while the computation demands only, so to speak, that of the numbers α, β, γ, δ, ϵ, . . . which is very easy.

As to the values of $\Delta u_0, \Delta^2 u_0, \ldots \Delta^5 u_0, \ldots$ the equations (1), (2), . . . (5) give each of them in terms of the immediate differences $\Delta, \Delta', \ldots \Delta^{\text{IV}} \ldots$

33. This was not the form in which the first solution of our problem was given by Newton in Lemma V. of the third book of his *Principia.* Laplace, in his *Mécanique Céleste,* has given a solution virtually the same as Newton's, but in a rather different form, which we will now demonstrate.

We will, in the first instance, establish an *Algebraical Lemma,* which for the sake of symmetry we shall find it convenient to use in certain subtractions, and which may prove useful on other occasions.

"Given any number, m, of binomial factors, as $x-a$ or α, $x-b$ or β, $x-c$ or γ, &c. . . . ; and the same number involving y instead of x, but with the same second terms, viz., $y-a$ or A, $y-b$ or B, $y-c$ or C . . . , the difference $\alpha\beta\gamma \ldots -\text{ABC} \ldots$ will be divisible by $x-y$; and the quotient will have two equivalent symmetrical forms."

1°. We see at once that the difference will be a polynomial of the form

$$x^m-y^m-\text{P}(x^{m-1}-y^{m-1})+\text{Q}(x^{m-2}-y^{m-2})-\text{R}(x^{m-3}-y^{m-3})\ldots\pm\text{W}.(x-y),$$

and that this polynomial is divisible by $x-y$;

2°. If we suppose $x-y=z$, it is easy to see that the differences $\alpha-\text{A}$, $\beta-\text{B}$, $\gamma-\text{C}$, ... are each equal to z, and that we shall thus have, *ad libitum,*

$$\begin{array}{lll} \text{either (1)} & \alpha=\text{A}+z, \quad \beta=\text{B}+z, \quad \gamma=\text{C}+z\ldots, \\ \text{or (2)} & \text{A}=\alpha-z, \quad \text{B}=\beta-z, \quad \text{C}=\gamma-z\ldots, \end{array}$$

Thus, according as we employ the relations (1) or (2), and take m equal to 2, 3, 4, . . . , we shall have the following as the differences of the products under consideration:—

Successive values of	According to the relations (1).		According to the relations (2).
$\alpha\beta \quad -\text{AB}$	$=(\text{B}+\alpha)z$	$=\pi z$	$=(\beta \ +\text{A})z,$
$\alpha\beta\gamma \quad -\text{ABC}$	$=(\text{C}\pi+\alpha\beta)z$	$=\pi_1 z$	$=(\gamma\pi \ +\text{AB})z,$
$\alpha\beta\gamma\delta \quad -\text{ABCD}$	$=(\text{D}\pi_1+\alpha\beta\gamma)z$	$=\pi_2 z$	$=(\delta\pi_1+\text{ABC})z,$
$\alpha\beta\gamma\delta\epsilon-\text{ABCDE}$	$=(\text{E}\pi_2+\alpha\beta\gamma\delta)z$	$=\pi_3 z$	$=(\epsilon\pi_2+\text{ABCD})z,$
. . . . .	. . . . .	. . .	. . . . .

We have thus obtained two sets of values for the quotients π, π_1, π_2, π_3, . . . the expansion and comparison of which will give the following symmetrical relations:—

$$\begin{aligned}
B + \alpha &= \pi = \beta + A,\\
BC + C\alpha + \alpha\beta &= \pi_1 = \beta\gamma + \gamma A + AB,\\
BCD + CD\alpha + D\alpha\beta + \alpha\beta\gamma &= \pi_2 = \beta\gamma\delta + \gamma\delta A + \delta AB + ABC,\\
BCDE + CDE\alpha + DE\alpha\beta + E\alpha\beta\gamma + \alpha\beta\gamma\delta &= \pi_3 = \beta\gamma\delta\epsilon + \gamma\delta\epsilon A + \delta\epsilon AB\\
&\qquad + \epsilon ABC + ABCD,
\end{aligned}$$

. . . .

and it may be observed that either of these two series may be derived from the other, if in that which we assume as given, we write $\alpha, \beta, \gamma, \ldots$ for A, B, C, ..., respectively, and A, B, C, ... for $\alpha, \beta, \gamma, \ldots$, respectively.

34. So much being premised, let $n, p, q, r, s, \ldots$ be values of x to which correspond the following given values of u_x, viz., $u, u', u'', u''', u^{\text{iv}} \ldots$ It is clear that we may assume

$$\begin{aligned}
u_x = u + B(x-n) + C(x-n)(x-p) + D(x-n)(x-p)(x-q)\\
+ E(x-n)(x-p)(x-q)(x-r) + \ldots,
\end{aligned}$$

where the coefficients B, C, D, E, ... are to be determined by means of the given values of u_x.

If we make $n=0$ in the above value of u_x, it becomes identical with the one in art. 29; but we have preferred the above form, because it leads to more symmetrical results.

Now putting x equal to $n, p, q \ldots$ successively, we have

$$\begin{aligned}
u &= u,\\
u' &= u + B(p-n),\\
u'' &= u + B(q-n) + C(q-n)(q-p),\\
u''' &= u + B(r-n) + C(r-n)(r-p) + D(r-n)(r-p)(r-q),\\
u^{\text{iv}} &= u + B(s-n) + C(s-n)(s-p) + D(s-n)(s-p)(s-q)\\
&\qquad + E(s-n)(s-p)(s-q)(s-r),
\end{aligned}$$

. . . .

From these equations we can deduce by means of the lemma in art. 33,

$$\begin{aligned}
u' - u &= B(p-n),\\
u'' - u' &= B(q-p) + C(q-n)(q-p),\\
u''' - u'' &= B(r-q) + C(r-q).[(q-n) + (r-p)] + D(r-n)(r-p)(r-q),\\
u^{\text{iv}} - u''' &= B(s-r) + C(s-r).[(r-p) + (s-n)]\\
&\qquad + D(s-r)[(r-n)(r-p) + (r-n)(s-q) + (s-p)(s-q)]\\
&\qquad + E(s-n)(s-p)(s-q)(s-r).
\end{aligned}$$

. . . .

Now dividing and using the symbol δ for brevity, we obtain

$$\begin{aligned}
\frac{u'-u}{p-n} &= \delta u = B,\\
\frac{u''-u'}{q-p} &= \delta u' = B + C(q-n),
\end{aligned}$$

$$\frac{u'''-u''}{r-q}=\delta u''=B+C(q-n)+C(r-p)+D(r-n)(r-p),$$

$$\frac{u^{\text{IV}}-u'''}{s-r}=\delta u'''=B+C(r-p)+C(s-n)+D(r-n)(r-p)+D(r-n)(s-q)$$
$$+D(s-p)(s-q)+E(s-n)(s-p)(s-q).$$

. . . .

A similar operation will give also

$$\frac{\delta u'-\delta u}{q-n}=\delta^2 u=C,$$

$$\frac{\delta u''-\delta u'}{r-p}=\delta^2 u'=C+D(r-n),$$

$$\frac{\delta u'''-\delta u''}{s-q}=\delta^2 u''=C+D(r-n)+D(s-p)+E(s-n)(s-p),$$

. . . .

and in the same way we shall find

$$\frac{\delta^2 u'-\delta^2 u}{r-n}=\delta^3 u=D,$$

$$\frac{\delta^2 u''-\delta^2 u'}{s-p}=\delta^3 u'=D+E(s-n),$$

. . . .

$$\frac{\delta^3 u'-\delta^3 u}{s-n}=\delta^4 u=E.$$

. . . .

The other coefficients, if required, would be found in exactly the same way.

Substituting now the values of the four first coefficients, as just found, we have for the value of u_x

$$u_x=u+(x-n).\delta u+(x-n)(x-p).\delta^2 u+(x-n)(x-p)(x-q).\delta^3 u$$
$$+(x-n)(x-p)(x-q)(x-r).\delta^4 u+\&c.;$$

The process of computation by means of this formula is very easy, the numerical values of the quantities δu, $\delta^2 u$, . . . being known, and being in most instances sufficiently small to allow of our confining ourselves to a few of the first terms of the series; a point easily decided in any case. This is the form adopted by Laplace, *Mécanique Céleste*, tom. i., p. 200.

35. Notwithstanding the dissimilarity between this solution and ours, it may be shown that they agree exactly; and we will conclude this paper by proving that such is the case.

If we substitute for δu, $\delta^2 u$, $\delta^3 u$, . . . their values deduced from the mode of their formation, as explained above, we shall find

$$\delta u = \frac{u'}{p-n} + \frac{u}{n-p},$$

$$\delta^2 u = \frac{u''}{(q-n)(q-p)} + \frac{u'}{(p-n)(p-q)} + \frac{u}{(n-p)(n-q)},$$

$$\delta^3 u = \frac{u'''}{(r-n)(r-p)(r-q)} + \frac{u''}{(q-n)(q-p)(q-r)} + \frac{u'}{(p-n)(p-q)(p-r)} + \frac{u}{(n-p)(n-q)(n-r)},$$

. . . .

Nothing can be more symmetrical than these expressions.

Now in order to compare this solution with ours, we must, as already observed, make $n=0$; and we shall then have, for example,

$$\delta^3 u = D = \frac{u'''}{r(r-p)(r-q)} + \frac{u''}{q(q-p)(q-r)} + \frac{u'}{p(p-q)(p-r)} - \frac{u}{pqr},$$

which, for brevity, we will write

$$D = \frac{u'''}{R} + \frac{u''}{Q} + \frac{u'}{P} - \frac{u}{pqr}.$$

But in art. 29 we found the following value for D, which ought to be equal to the preceding, since we have put $n=0$,

$$D = \frac{\Delta''}{R} + \frac{\Delta'}{Q} + \frac{\Delta}{P},$$

or

$$D = \frac{u'''-u}{R} + \frac{u''-u}{Q} + \frac{u'-u}{P}.$$

We must therefore have

$$\frac{1}{R} + \frac{1}{Q} + \frac{1}{P} = \frac{1}{pqr},$$

which is seen to be the case, when for R, Q, P, we substitute their values.

APPENDIX.

On a case that may occur in astronomical interpolations.

I. When a series of astronomical observations is made, the element observed may be considered as a function of the time, since it varies therewith. Let t denote the time. We may then, as in the calculus of finite differences, represent the value of the element under consideration, whether observed or computed, by u_t; the time t being considered as the independent variable.

Suppose that we know n values of the element, or of the function u_t, viz., a, a', a'', . . . corresponding to the times θ, θ', θ'', . . . which are supposed to be successive and equidistant, and that we wish to deduce a general expression for u_t that shall be true for any value of t not too remote from the period under consideration. The required solution will be possible, and indeed very simple, if the successive differences of the given

values become either constant, or small enough to admit of our neglecting all beyond a certain order.

Suppose, for example, that this is the case with the fourth differences, so that we may neglect the differences of the fifth and higher orders: it will be sufficient to consider five values of u_t, or to have $n=5$.

Also let

$$\theta'-\theta=\theta''-\theta'=\theta'''-\theta''=\theta^{\text{iv}}-\theta'''=\alpha.$$

The common difference of the increasing arithmetical progression $\theta, \theta', \ldots \theta^{\text{iv}}$, will then be α, and it is clear that we may assume

$$u_t=\mathrm{A}+\mathrm{B}(t-\theta)+\mathrm{C}(t-\theta)(t-\theta')+\mathrm{D}(t-\theta)(t-\theta')(t-\theta'')$$
$$+\mathrm{E}(t-\theta)(t-\theta')(t-\theta'')(t-\theta''').$$

The given values of u_t will then lead to the following equations:—

$$\begin{aligned}
a &= \mathrm{A},\\
a' &= a+\mathrm{B}.\alpha,\\
a'' &= a+\mathrm{B}.2\alpha+\mathrm{C}.2\alpha^2,\\
a''' &= a+\mathrm{B}.4\alpha+\mathrm{C}.6\alpha^2+\mathrm{D}.6\alpha^3,\\
a^{\text{iv}} &= a+\mathrm{B}.4\alpha+\mathrm{C}.12\alpha^2+\mathrm{D}.24\alpha^3+\mathrm{E}.24\alpha^4,
\end{aligned}$$

and we may thence conclude almost directly, from the known values of the *differences* of any order in terms of given *numbers*, that

$$\mathrm{A}=a;\ \mathrm{B}=\frac{1}{\alpha}\cdot\delta a;\ \mathrm{C}=\frac{1}{1.2.\alpha^2}\cdot\delta^2 a;\ \mathrm{D}=\frac{1}{1.2.3.\alpha^3}\cdot\delta^3 a;\ \mathrm{E}=\frac{1}{1.2.3.4.\alpha^4}\delta^4 a.$$

We shall thus have for the solution sought—

$$(h)\quad u_t=a+\frac{t-\theta}{\alpha}\cdot\delta a+\frac{(t-\theta)(t-\theta')}{1.2.\alpha^2}\cdot\delta^2 a+\frac{(t-\theta)(t-\theta')(t-\theta'')}{1.2.3.\alpha^3}\cdot\delta^3 a$$
$$+\frac{(t-\theta)(t-\theta')(t-\theta'')(t-\theta''')}{1.2.3.4.\alpha^4}\cdot\delta^4 a.$$

It is also easy from the symmetry to perceive the form this formula would take if n were larger, and we were called upon to consider differences of a higher order than the fourth; and we see readily what terms should then be added to the preceding.

II. Now if we desire to reckon the time t from the middle of the period under consideration, so that $\theta''=0$, and to take the time between two consecutive observations as the unit of time, we must, in the formula (h), put

$$\alpha=1,\quad \theta=-2,\quad \theta'=-1,\quad \theta''=0,\quad \theta'''=+1.$$

That formula will then give

$$(h')\qquad u_t=a+\frac{t+2}{1}\cdot\delta a+\frac{(t+2)(t+1)}{1.2}\cdot\delta^2 a+\frac{(t+2)(t+1)t}{1.2.3}\cdot\delta^3 a$$
$$+\frac{(t+2)(t+1)t(t-1)}{1.2.3.4}\cdot\delta^4 a,$$

where, as before, we have only to do with a and its differences, as found from the five given values of u_t.

III. Stirling (followed by Lacroix, tome iii., pp. 27–8) has given the formula for a similar case under a different form, which he has only been able to generalize by induction. M. Bessel, who has employed the formula

given by Stirling, appears to have demonstrated it in the following manner for the case where $n=5$.

Let $$u_t = \mathrm{A} + \mathrm{B}t + \mathrm{C}t^2 + \mathrm{D}t^3 + \mathrm{E}t^4.$$

Then making t successively equal to -2, -1, 0, $+1$ and $+2$, we shall have five equations which we may denote respectively by (-2), (-1), (0), $(+1)$ and $(+2)$.

Then
$$(+1)+(-1) \text{ gives } \frac{a'''+a'}{2} - a'' = \mathrm{C}+\mathrm{E};$$
$$(+2)+(-2) \quad ,, \quad \frac{a^{\text{iv}}+a}{2} - a'' = 4\mathrm{C}+16\mathrm{E};$$
$$(0) \quad ,, \quad a'' = \mathrm{A};$$
$$(+1)-(-1) \quad ,, \quad \frac{a'''-a'}{2} = \mathrm{B}+\mathrm{D};$$
$$(+2)-(-2) \quad ,, \quad \frac{a^{\text{iv}}-a}{2} = 2\mathrm{B}+8\mathrm{D}.$$

We hence obtain, by means of the well known values of the various orders of differences in terms of the given numbers a, a', ... a^{iv}, the following values:—
$$\mathrm{A}=a'', \quad \mathrm{B}=\frac{\delta a'+\delta a''}{2} - \frac{\delta^3 a+\delta^3 a'}{2.6}, \quad \mathrm{C}=\frac{\delta^2 a'}{2} - \frac{\delta^4 a}{24},$$
$$\mathrm{D}=\frac{\delta^3 a+\delta^3 a'}{2.6}, \quad \mathrm{E}=\frac{\delta^4 a}{24}.$$

Substituting these values, and rearranging the terms with reference to the powers of t, we get
$$(g) \quad u_t = a'' + \left[\frac{\delta a'+\delta a''}{2}\right]\cdot\frac{t}{1} + (\delta^2 a')\cdot\frac{t^2}{1.2} + \left[\frac{\delta^3 a+\delta^3 a'}{2}\right]\cdot\frac{t(t^2-1)}{1.2.3}$$
$$+ (\delta^4 a)\cdot\frac{t^2(t^2-1)}{1.2.3.4}.$$

But this formula is subject to the inconvenience of involving differences of several of the given values; while the formula (h') contains only the several differences of the first value a. It also appears much less symmetrical, and less easy to remember: and, above all, it is more difficult of rigorous generalization than the formulæ (h) and (h').

Notwithstanding the apparent difference of the formulæ (h') and (g), it is easy to prove that they only give the same result under different forms.

Another case of astronomical interpolation.

IV. In astronomy the third (if not the second) differences are generally constant, or, at all events, small enough to permit of our neglecting the fourth differences; and the formula (h) then reduces to
$$u_t = a + \frac{t-\theta}{\alpha}\cdot\delta a + \frac{(t-\theta)(t-\theta')}{1.2.\alpha^2}\cdot\delta^2 a + \frac{(t-\theta)(t-\theta')(t-\theta'')}{1.2.3.\alpha^3}\cdot\delta^3 a,$$
in which θ, θ'; θ'', θ''', being equidistant values of t, corresponding to the known values a, a', a'', a''', of u_t, we have
$$\alpha = \theta' - \theta = \theta'' - \theta' = \theta''' - \theta''.$$

Now, if the third differences are constant, we take from the tables, or we know from the observations, only the four values a, a', a'', a''', of which

the former two will correspond to times that precede, and the others to times that follow the epoch t for which we are computing, and which must consequently fall between θ' and θ''. It will therefore now be convenient to make $\theta'=0$, whence

$$\theta=-a, \quad \text{and } \theta''=+a.$$

The formula then gives

$$(h'') \qquad u_t=a+\frac{t+a}{a}\cdot\delta a+\frac{(t+a)t}{1.2.a^2}\cdot\delta^2 a+\frac{t(t^2-a^2)}{1.2.3.a^3}\cdot\delta^3 a;$$

or,

$$(l)\left\{\begin{aligned} u_t&=a+\delta a+\frac{t}{a}[\delta a+\delta^2 a]+\frac{t^2-at}{1.2.a^2}\cdot\delta^2 a+\frac{t(t^2-a^2)}{1.2.3a^3}\cdot\delta^3 a\\ &=a'+\frac{t}{a}\cdot\delta a'+\frac{t(t-a)}{a^2}\cdot\frac{1}{4}[\delta^2 a'+\delta^2 a]+\frac{t(t-a)}{1.2.a^2}\left[\frac{t+a}{3a}-\frac{1}{2}\right]\cdot\delta^3 a\\ &=a'+\frac{t}{a}\cdot\delta a'+\frac{t(t-a)}{a^2}\cdot\frac{1}{4}[\delta^2 a'+\delta^2 a]+\frac{t(t-a)(t-\frac{1}{2}a)}{1.2.3.a^3}\cdot\delta^3 a.\end{aligned}\right.$$

This formula (l) coincides exactly with formula (2) of p. 99 of *L'Astronomie pratique* of *M. Francœur*, when we make

$$a=12^h, \text{ whence } \frac{1}{2}a=6^h,$$

and put $\Delta'=\delta a'$, and $\phi=\frac{1}{4}(\delta^2 a'+\delta^2 a)$.

It may also be made to coincide with the formula (3) at the top of p. 100 of the same work. In fact, the correction x or u_t-a', to be applied to the value a', becomes

$$(n) \qquad x=\frac{t}{a}\left[\delta a'-\phi+\frac{1}{12}\cdot\delta^3 a\right]+\frac{t^2}{a^2}\left[\phi-\frac{1}{4}\cdot\delta^3 a\right]+\frac{t^3}{a^3}\cdot\frac{1}{6}\cdot\delta^3 a,$$

which is of the following form—

$$x=\mathrm{A}\cdot\frac{t}{a}+\mathrm{B}\cdot\frac{t^2}{a^2}+\mathrm{C}\cdot\frac{t^3}{a^3}.$$

If now we pursue the reverse process and start with this formula, we shall find the values of A, B, and C, by making t equal to $-a$, a, $2a$ successively in the equation

$$u_t=a'+\mathrm{A}\cdot\frac{t}{a}+\mathrm{B}\cdot\frac{t^2}{a^2}+\mathrm{C}\cdot\frac{t^3}{a^3}.$$

We thus get

$$\begin{aligned} a&=a'-\mathrm{A}+\mathrm{B}-\mathrm{C},\\ a''&=a'+\mathrm{A}+\mathrm{B}+\mathrm{C},\\ a'''&=a'+2\mathrm{A}+4\mathrm{B}+8\mathrm{C}.\end{aligned}$$

The two first of these equations give

$$a''+a-2a'=2\mathrm{B},$$

whence $\qquad \mathrm{B}=\frac{1}{2}\cdot\delta^2 a=\frac{1}{4}(\delta^2 a'+\delta^2 a)-\frac{1}{4}\cdot\delta^3 a.$

Also $\qquad a''-a=2(\mathrm{A}+\mathrm{C}).$

But, from the third, $\qquad a'''-4\mathrm{B}-a'=2\mathrm{A}+8\mathrm{C};$

whence by subtraction,

$$a'''-a''-(a'-a)-4B=6C,$$

and
$$\delta a''-\delta a-2.\delta^2 a=6C;$$

or, since
$$\delta a''-\delta a=\delta^2 a'+\delta^2 a=2.\delta^2 a+\delta^3 a,$$

we have
$$C=\frac{\delta^3 a}{1.2.3}.$$

Finally, as the second equation gives

$$A=a''-a'-B-C,$$

we shall have
$$A=\delta a'-\phi+\frac{1}{12}\cdot\delta^3 a,$$

and substituting these values of A, B, C, we get the formula (n).

It may be noticed that in using the formula (l), if we neglect altogether the last term which involves $\delta^3 a$, the second differences would still be partly corrected by the differences of the third order. In fact, the formula would then give

$$u_t=a'+\frac{t}{\alpha}\cdot\delta a'+\frac{t(t-\alpha)}{\alpha^2}\cdot\phi.$$

But,

$$\phi=\frac{1}{4}(\delta^2 a'+\delta^2 a)=\frac{1}{2}\left(\delta^2 a+\frac{1}{2}\delta^3 a\right)=\frac{1}{2}\left(\delta^2 a'-\frac{1}{2}\delta^3 a\right),$$

so that we shall have

$$u_t=a'+\frac{t}{\alpha}\cdot\delta a'+\frac{t(t-\alpha)}{1.2.\alpha^2}\left(\delta^2 a-\frac{1}{2}\delta^3 a\right).$$

It is from this last term that M. Mathieu has computed a table inserted in the *Connaissance des Temps.* Supposing $\alpha=12^h$, and that the motion of the star is uniform during those 12 hours, the two first terms $a'+\frac{t}{12^h}\cdot\delta a'$ are given; and we then apply the correction resulting from the last term by means of the table in question, which is computed for values of t differing by 10 minutes.

Generally speaking, in all these formulæ, we give to α such integral values as we may require, making it most frequently equal to 12^h, 6^h, 3^h, or 1^h.

On the Value of Apportionable (or Complete) Annuities. (Continued.)

By Thomas B. Sprague, M.A., *Actuary of the Equity and Law Life Assurance Society, and Vice-President of the Institute of Actuaries.*

Having thus found the required formula for the value of $\mathbf{a}^{(m)}_{k\overline{|}\tau}$, viz.:—

$$\mathbf{a}^{(m)}_{k\overline{|}\tau}=a_k+\frac{m+1}{2m}-\tau-\frac{\mu+\delta}{12m^2}(m^2-1+6m\tau-6m^2\tau^2)$$
$$-\frac{1}{12m^2}\frac{D''_k}{D_k}\tau(1-m\tau)(1-2m\tau)$$

we are now in a position to proceed with the transformation of the expression found in the last Number of this *Journal* (vol. xiii., p. 378), viz.:—

$$\frac{r}{n}v^{\frac{1}{mn}}\mathrm{a}^{(m)}_{k\,\overline{\frac{r-1}{mn}}|}-\frac{r}{n}\mathrm{a}^{(m)}_{k\,\overline{\frac{r}{mn}}|}.$$

In fact, making $\tau=\dfrac{r-1}{mn}$, and $\dfrac{r}{mn}$, successively, we get

$$\begin{aligned}\frac{r}{n}v^{\frac{1}{mn}}\mathrm{a}^{(m)}_{k\,\overline{\frac{r-1}{mn}}|}-\frac{r}{n}\mathrm{a}^{(m)}_{k\,\overline{\frac{r}{mn}}|}=\frac{r}{n}v^{\frac{1}{mn}}\left\{a_k+\frac{m+1}{2m}-\frac{r-1}{mn}-\frac{\mu+\delta}{12m^2}\left[m^2-1+6\frac{r-1}{n}-6\frac{(r-1)^2}{n^2}\right]-\frac{1}{12m^2}\frac{\mathrm{D}'_k}{\mathrm{D}_k}\frac{r-1}{mn}\left(1-\frac{r-1}{n}\right)\left(1-2\frac{r-1}{n}\right)\right\}\\ -\frac{r}{n}\left\{a_k+\frac{m+1}{2m}-\frac{r}{mn}-\frac{\mu+\delta}{12m^2}\left(m^2-1+\frac{6r}{n}-\frac{6r^2}{n^2}\right)-\frac{1}{12m^2}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\cdot\frac{r}{mn}\left(1-\frac{r}{n}\right)\left(1-\frac{2r}{n}\right)\right\}.\end{aligned}$$

This expression has now to be summed with respect to r, giving it the values, 1, 2, 3, ... n; and then n has to be supposed infinite. Rearranging the terms, we have

$$\begin{aligned}\Sigma\left\{\frac{r}{n}v^{\frac{1}{mn}}\mathrm{a}^{(m)}_{k\,\overline{\frac{r-1}{mn}}|}-\frac{r}{n}\mathrm{a}^{(m)}_{k\,\overline{\frac{r}{mn}}|}\right\}=-(1-v^{\frac{1}{mn}})\left\{\left(a_k+\frac{m+1}{2m}\right)\Sigma\frac{r}{n}-\frac{1}{m}\Sigma\frac{r^2}{n^2}-\frac{\mu+\delta}{12m^2}\left[(m^2-1)\Sigma\frac{r}{n}+6\Sigma\frac{r^2}{n^2}-6\Sigma\frac{r^3}{n^3}\right]-\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\left[\Sigma\frac{r^2}{n^2}-3\Sigma\frac{r^3}{n^3}+2\Sigma\frac{r^4}{n^4}\right]\right\}\\ +v^{\frac{1}{mn}}\left\{\frac{1}{m}\Sigma\frac{r}{n^2}+\frac{\mu+\delta}{2m^2}\left[\Sigma\frac{r}{n^2}-2\Sigma\frac{r^2}{n^3}+\Sigma\frac{r}{n^3}\right]+\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\left[\Sigma\frac{r}{n^2}-6\Sigma\frac{r^2}{n^3}+3\Sigma\frac{r}{n^3}+6\Sigma\frac{r^3}{n^4}-6\Sigma\frac{r^2}{n^4}+2\Sigma\frac{r}{n^4}\right]\right\}\quad\ldots(31)\end{aligned}$$

But since $\delta=-\log_e v$, or $v=e^{-\delta}$,

we have
$$v^{\frac{1}{mn}}=e^{-\frac{\delta}{mn}}=1-\frac{\delta}{mn}+\frac{\delta^2}{2m^2n^2}-\frac{\delta^3}{6m^3n^3}+\ldots.$$

and
$$1-v^{\frac{1}{mn}}=\frac{\delta}{mn}\left(1-\frac{\delta}{2mn}+\frac{\delta^2}{6m^2n^2}-\ldots.\right)$$

Also
$$\Sigma\frac{r}{n}=\frac{1}{n}\frac{n(n+1)}{2}=\frac{n+1}{2}$$

$$\Sigma\frac{r^2}{n^2}=\frac{n(n+1)(2n+1)}{6n^2}=\frac{n}{3}+\tfrac{1}{2}+\frac{1}{6n}$$

$$\Sigma\frac{r^3}{n^3}=\frac{n^2(n+1)^2}{4n^3}=\frac{n}{4}+\tfrac{1}{2}+\frac{1}{4n}$$

$$\Sigma\frac{r^4}{n^4}=\frac{n}{5}+\tfrac{1}{2}+\frac{1}{3n}-\frac{1}{30n^2}.$$ (See vol. xi., p. 203.)

Then proceeding to the limit by making n infinite, and using the abbreviation "Lt" to denote "the limiting value, when n is supposed to be infinite, of," we have

$$\text{Lt}(1-v^{\frac{1}{mn}})\Sigma\frac{r}{n}=\text{Lt}\frac{\delta}{mn}\left(1-\frac{\delta}{2mn}+\ldots\right)\left(\frac{n}{2}+\tfrac{1}{2}\right)$$

$$=\text{Lt}\frac{\delta}{m}\left(1-\frac{\delta}{2mn}+\ldots\right)\left(\tfrac{1}{2}+\frac{1}{2n}\right)=\frac{\delta}{2m}$$

$$\text{Lt}(1-v^{\frac{1}{mn}})\Sigma\frac{r^2}{n^2}=\text{Lt}\frac{\delta}{mn}\left(1-\frac{\delta}{2mn}+\ldots\right)\left(\frac{n}{3}+\tfrac{1}{2}+\frac{1}{6n}\right)$$

$$=\text{Lt}\frac{\delta}{m}\left(1-\frac{\delta}{2mn}+\ldots\right)\left(\tfrac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right)=\frac{\delta}{3m}$$

$$\text{Lt}(1-v^{\frac{1}{mn}})\Sigma\frac{r^3}{n^3}=\text{Lt}\frac{\delta}{mn}\left(1-\frac{\delta}{2mn}+\ldots\right)\left(\frac{n}{4}+\tfrac{1}{2}+\frac{1}{4n}\right)=\frac{\delta}{4m}$$

$$\text{Lt}(1-v^{\frac{1}{mn}})\Sigma\frac{r^4}{n^4}=\text{Lt}\frac{\delta}{mn}\left(1-\frac{\delta}{2mn}+\ldots\right)\left(\frac{n}{5}+\tfrac{1}{2}+\frac{1}{3n}-\frac{1}{30n^2}\right)=\frac{\delta}{5m}$$

Also, when n is infinite, $v^{\frac{1}{mn}}$ becomes equal to unity.

And in the same case

$$\text{Lt}\,\Sigma\frac{r}{n^2}=\text{Lt}\left(\tfrac{1}{2}+\frac{1}{2n}\right)=\tfrac{1}{2}$$

$$\text{Lt}\,\Sigma\frac{r^2}{n^3}=\text{Lt}\left(\tfrac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right)=\tfrac{1}{3}$$

$$\text{Lt}\,\Sigma\frac{r^3}{n^4}=\text{Lt}\left(\tfrac{1}{4}+\frac{1}{2n}+\frac{1}{6n^2}-\frac{1}{30n^3}\right)=\tfrac{1}{4}.$$

We also see that

$$\text{Lt}\,\Sigma\frac{r}{n^3}=0,\quad \text{Lt}\,\Sigma\frac{r^2}{n^4}=0,\quad \text{Lt}\,\Sigma\frac{r}{n^4}=0.$$

Substituting these limiting values, we get the limit of the value of

$$\Sigma\left\{\frac{r}{n}v^{\frac{1}{mn}}\mathrm{a}^{(m)}_{k\overline{\left|\frac{r-1}{mn}\right.}}-\frac{r}{n}\mathrm{a}^{(m)}_{k\overline{\left|\frac{r}{n}\right.}}\right\}$$

when n is made infinite

$$=-\left\{a_k+\frac{m+1}{2m}\right\}\frac{\delta}{2m}+\frac{1}{m}\cdot\frac{\delta}{3m}+\frac{\mu+\delta}{12m^2}\left\{(m^2-1)\frac{\delta}{2m}+6\frac{\delta}{3m}-6\frac{\delta}{4m}\right\}$$

$$-\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\left\{\frac{\delta}{3m}-3\frac{\delta}{4m}+2\frac{\delta}{5m}\right\}$$

$$+\frac{1}{m}\cdot\tfrac{1}{2}+\frac{\mu+\delta}{2m^2}\{\tfrac{1}{2}-\tfrac{2}{3}\}+\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\{\tfrac{1}{2}-\tfrac{6}{3}+\tfrac{6}{4}\}$$

$$=\frac{1-\delta a_k}{2m}-\frac{\mu}{12m^2}-\frac{\delta}{4m}+\frac{(\mu+\delta)\delta}{24m}-\frac{\delta}{720m^4}\cdot\frac{\mathrm{D}''_k}{\mathrm{D}_k}\quad\ldots\ldots\quad(32)$$

and this ought therefore to be the value of the correction when the annuity is payable to the day of death.

Testing this result, however, by applying it to the case of a perpetual annuity certain, and making accordingly $\mu=0$, $\frac{\mathrm{D}''_k}{\mathrm{D}_k}=\delta^2$, $a_k=\frac{1}{i}$, it becomes

$$\frac{1}{2m}\left\{1-\frac{\delta}{i}-\frac{\delta}{2}+\frac{\delta^2}{12}-\frac{\delta^3}{360m^3}\right\}=\frac{1}{2m}\left\{\frac{\delta^4}{720}-\frac{\delta^3}{360m^3}\right\}\text{ nearly.}$$

The value in this case ought to vanish, and we thus see that the last term in (32) is not correct. The reason of this is, that in proceeding to the limit of (31), our process takes account of the small quantities of the third order (*i.e.* terms involving δ^3, $\frac{\delta\mathrm{D}''_k}{\mathrm{D}_k}$, &c.) in the first term, but not of the similar quantities in the second term. Instead of (32) we can only make use of the formula obtained by omitting the inaccurate last term; and indeed the above test shows that the correct formula will contain no term of the third order, so that the formula for the value of the correction will be,—accurately as far as terms of the third order inclusive

$$\frac{1-\delta a_k}{2m}-\frac{\mu}{12m^2}-\frac{\delta}{4m}+\frac{(\mu+\delta)\delta}{24m},$$

which is the same as we have already found (22). (Vol. xiii., p. 377.)

As a still further test of the accuracy of our results, let us now suppose that money bears no interest or $\delta=0$; then for a_k we must

substitute e_k, the curtate expectation of life, and the formula (23) gives as the value of the *complete expectation*, $e_k+\frac{1}{2}-\frac{\mu}{12}$, which is independent of m, as of course it ought to be, and agrees with the value found by Mr. Woolhouse (vol. xi. p. 328).

Since writing the preceding, I have noticed that Griffith Davies uses the phrase "complete annuity" to denote the annuity payable up to the day of death. This is analagous to the common phrase "complete expectation of life"; and seems preferable to the term I have hitherto employed in this paper.

The next problem which suggests itself is to find the value of an apportionable (or a complete) annuity payable m times a year, when the first payment, instead of being due at the time $\frac{1}{m}$, is due at the time τ, where τ is less than $\frac{1}{m}$.

Let each m-part be subdivided into n equal portions, and suppose $\tau=\frac{\sigma}{mn}$, where σ is supposed to be an integer. Whatever the value of τ, this can be done to any degree of accuracy required, by taking n and σ sufficiently large.

Suppose further that if death occur in the subdivisions numbered respectively

$$1,\quad 2,\quad 3,\ldots\ \sigma,\ \sigma+1,\ \sigma+2,\ldots\ n,\quad 1,\quad 2,\ldots\ \sigma,\quad \sigma+1,\ldots$$

there is payable a sum

$$\frac{n-\sigma+1}{mn},\ \frac{n-\sigma+2}{mn},\ldots\frac{n}{mn},\ \frac{1}{mn},\ \frac{2}{mn},\ldots\ \frac{n-\sigma}{mn},\ \frac{n-\sigma+1}{mn},\ldots\ldots\frac{n}{mn},\ \frac{1}{mn},\ldots$$

then by supposing n infinite, we pass to the conditions of the problem proposed.

If now $\frac{\sigma}{mn}$ be added to each of the above payments, they become

$$\frac{n+1}{mn},\quad \frac{n+2}{mn},\ldots\frac{n+\sigma}{mn},\frac{\sigma+1}{mn},\frac{\sigma+2}{mn},\ldots\frac{n}{mn},\quad \frac{n+1}{mn},\ldots\ldots\frac{n+\sigma}{mn},\frac{\sigma+1}{mn},\ldots$$

$$\text{or}\quad \left.\begin{matrix}\frac{1}{mn}\\+\frac{1}{m}\end{matrix}\right\},\quad \left.\begin{matrix}\frac{2}{mn}\\+\frac{1}{m}\end{matrix}\right\},\ldots\left.\begin{matrix}\frac{\sigma}{mn}\\+\frac{1}{m}\end{matrix}\right\},\frac{\sigma+1}{mn},\ \frac{\sigma+2}{mn},\ldots\frac{n}{mn},\quad \left.\begin{matrix}\frac{1}{mn}\\+\frac{1}{m}\end{matrix}\right\},\ldots\ldots\left.\begin{matrix}\frac{\sigma}{mn}\\+\frac{1}{m}\end{matrix}\right\},\frac{\sigma+1}{mn},\ldots$$

Thus we see that the value of the correction is equal to the value of the correction already found (vol. xiii., p. 377)

+ the value of $\frac{1}{m}$ payable if death occur in one of the first σ subdivisions of any m-part

— the value of $\frac{\sigma}{mn}\left(=\tau\right)$ payable whenever death may occur.

$=\mathrm{P}+\mathrm{Q}-\mathrm{R}$, suppose.

Now to find the value of Q, we see that it is equal to

$$\frac{1}{m}\left\{v^{\frac{1}{mn}}(1-p_{k,\frac{1}{mn}})+v^{\frac{2}{mn}}(p_{k,\frac{1}{mn}}-p_{k,\frac{2}{mn}})+\ldots\ldots\ldots\ldots\ldots\ldots+v^{\frac{\sigma}{mn}}(p_{k,\frac{\sigma-1}{mn}}-p_{k,\frac{\sigma}{mn}})\right.$$
$$+v^{\frac{1}{m}+\frac{1}{mn}}(p_{k,\frac{1}{m}}-p_{k,\frac{1}{m}+\frac{1}{mn}})+v^{\frac{1}{m}+\frac{2}{mn}}(p_{k,\frac{1}{m}+\frac{1}{mn}}-p_{k,\frac{1}{m}+\frac{2}{mn}})+\ldots+v^{\frac{1}{m}+\frac{\sigma}{mn}}(p_{k,\frac{1}{m}+\frac{\sigma-1}{mn}}-p_{k,\frac{1}{m}+\frac{\sigma}{mn}})$$
$$+v^{\frac{2}{m}+\frac{1}{mn}}(p_{k,\frac{2}{m}}-p_{k,\frac{2}{m}+\frac{1}{mn}})+\ldots\ldots$$
$$\vdots$$
$$\left.+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right\}$$

$$=v^{\frac{1}{mn}}\left\{\mathrm{a}_k^{(m)}+\mathrm{a}^{(m)}_{k\overline{\left|\frac{1}{mn}\right.}}+\mathrm{a}^{(m)}_{k\overline{\left|\frac{2}{mn}\right.}}+\ldots\ldots\ldots+\mathrm{a}^{(m)}_{k\overline{\left|\frac{\sigma-1}{mn}\right.}}\right\}-\left\{\mathrm{a}^{(m)}_{k\overline{\left|\frac{1}{mn}\right.}}+\mathrm{a}^{(m)}_{k\overline{\left|\frac{2}{mn}\right.}}+\ldots\ldots\ldots+\mathrm{a}^{(m)}_{k\overline{\left|\frac{\sigma}{mn}\right.}}\right\}$$

$$=v^{\frac{1}{mn}}\left\{\mathrm{a}_k^{(m)}-\mathrm{a}^{(m)}_{k\overline{\left|\frac{\sigma}{mn}\right.}}\right\}-(1-v^{\frac{1}{mn}})\left\{\mathrm{a}^{(m)}_{k\overline{\left|\frac{1}{mn}\right.}}+\mathrm{a}^{(m)}_{k\overline{\left|\frac{2}{mn}\right.}}+\ldots\ldots+\mathrm{a}^{(m)}_{k\overline{\left|\frac{\sigma}{mn}\right.}}\right\}.$$

$$=v^{\frac{1}{mn}}\left\{a_k+\frac{m+1}{2m}-\frac{m^2-1}{12m^2}(\mu+\delta)-a_k-\frac{m+1}{2m}+\frac{\sigma}{mn}+\frac{\mu+\delta}{12m^2}\left(m^2-1+6\frac{\sigma}{n}-6\frac{\sigma^2}{n^2}\right)\right.$$
$$\left.+\frac{1}{12m^2}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\frac{\sigma}{mn}\left(1-\frac{\sigma}{n}\right)\left(1-2\frac{\sigma}{n}\right)\right\}$$
$$-(1-v^{\frac{1}{mn}})\left\{a_k+\frac{m+1}{2m}-\frac{1}{mn}-\frac{\mu+\delta}{12m^2}\left(m^2-1+6\frac{1}{n}-6\frac{1}{n^2}\right)-\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\left(\frac{1}{n}-3\cdot\frac{1}{n^2}+2\cdot\frac{1}{n^3}\right)\right.$$
$$+a_k+\frac{m+1}{2m}-\frac{2}{mn}-\frac{\mu+\delta}{12m^2}\left(m^2-1+6\cdot\frac{2}{n}-6\frac{2^2}{n^2}\right)-\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\left(\frac{2}{n}-3\frac{2^2}{n^2}+2\cdot\frac{2^3}{n^3}\right)$$
$$+\ \ldots\ldots$$
$$\vdots$$
$$\left.+a_k+\frac{m+1}{2m}-\frac{\sigma}{mn}-\frac{\mu+\delta}{12m^2}\left(m^2-1+6\frac{\sigma}{n}-6\frac{\sigma^2}{n^2}\right)-\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\left(\frac{\sigma}{n}-3\frac{\sigma^2}{n^2}+2\frac{\sigma^3}{n^3}\right)\right\}$$

$$\left.\begin{aligned}&=v^{\frac{1}{mn}}\left\{\frac{\sigma}{mn}+\frac{\mu+\delta}{2m^2}\left(\frac{\sigma}{n}-\frac{\sigma^2}{n^2}\right)+\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\frac{\sigma}{n}\left(1-\frac{\sigma}{n}\right)\left(1-\frac{2\sigma}{n}\right)\right\}\\&-(1-v^{\frac{1}{mn}})\left\{\sigma a_k+\sigma\frac{m+1}{2m}-\frac{\sigma(\sigma+1)}{2mn}-\frac{\mu+\delta}{12m^2}\left[\sigma(m^2-1)+\frac{3\sigma(\sigma+1)}{n}-\frac{\sigma(\sigma+1)(2\sigma+1)}{n^2}\right]\right.\\&\left.\qquad-\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\left[\frac{\sigma(\sigma+1)}{2n}-\frac{\sigma(\sigma+1)(2\sigma+1)}{2n^2}+\frac{\sigma^2(\sigma+1)^2}{2n^3}\right]\right\}\end{aligned}\right\}(33)$$

Now proceed to the limit (as above, p. 38), supposing n to become infinite, $\frac{\sigma}{n}$ being equal to $m\tau$.

Then
$$\mathrm{Lt}(1-v^{\frac{1}{mn}})\,\sigma=\mathrm{Lt}\frac{\sigma}{n}\cdot\frac{\delta}{m}\left(1-\frac{\delta}{2mn}+\ldots.\right)=\tau\delta$$
$$\mathrm{Lt}(1-v^{\frac{1}{mn}})\frac{\sigma(\sigma+1)}{n}=\mathrm{Lt}\frac{\delta}{m}\left(1-\frac{\delta}{2mn}+\ldots.\right)\frac{\sigma}{n}\left(\frac{\sigma}{n}+\frac{1}{n}\right)=\frac{\delta}{m}m^2\tau^2=m\delta\tau^2$$
$$\mathrm{Lt}(1-v^{\frac{1}{mn}})\frac{\sigma(\sigma+1)(2\sigma+1)}{n^2}=\mathrm{Lt}\frac{\delta}{m}\left(1-\frac{\delta}{2mn}+\ldots.\right)\frac{\sigma}{n}\left(\frac{\sigma}{n}+\frac{1}{n}\right)\left(2\frac{\sigma}{n}+\frac{1}{n}\right)=2\frac{\delta}{m}m^3\tau^3=2m^2\delta\tau^3$$
$$\mathrm{Lt}(1-v^{\frac{1}{mn}})\frac{\sigma^2(\sigma+1)^2}{n^3}=\mathrm{Lt}\frac{\delta}{m}\left(1-\frac{\delta}{2mn}+\ldots.\right)\frac{\sigma^2}{n^2}\left(\frac{\sigma}{n}+\frac{1}{n}\right)^2=\frac{\delta}{m}m^4\tau^4=m^3\delta\tau^4.$$

Hence the above expression (33) becomes
$$\tau+\frac{\mu+\delta}{2}\tau\left(\frac{1}{m}-\tau\right)+\frac{1}{12}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)$$
$$-\left\{\tau\delta\left(a_k+\frac{m+1}{2m}\right)-\frac{\delta\tau^2}{2}-\frac{\mu+\delta}{12m^2}[(m^2-1)\tau\delta+3m\delta\tau^2-2m^2\delta\tau^3]-\frac{1}{12m^3}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\left[\frac{m}{2}\delta\tau^2-m^2\delta\tau^3+\frac{1}{2}m^3\delta\tau^4\right]\right\}$$
$$=\tau+\frac{\mu+\delta}{2}\tau\left(\frac{1}{m}-\tau\right)+\frac{1}{12}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)-\tau\delta a_k-\tau\delta\frac{m+1}{2m}+\frac{\delta\tau^2}{2}+\frac{\delta(\mu+\delta)\tau}{12}\left(1-\frac{1}{m^2}+3\frac{\tau}{m}-2\tau^2\right)$$
$$=\tau(1-\delta a_k)+\frac{\mu}{2}\tau\left(\frac{1}{m}-\tau\right)+\frac{\tau\delta}{2}\left\{\frac{1}{m}-\tau-\frac{m+1}{m}+\tau\right\}+\frac{1}{12}\delta(\mu+\delta)\tau\left(1-\frac{1}{m^2}+3\frac{\tau}{m}-2\tau^2\right)+\frac{1}{12}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\tau\left(\frac{1}{m}-\tau\right)\left\{\frac{1}{m}-2\tau+\frac{\delta\tau}{2}\left(\frac{1}{m}-\tau\right)\right\}$$
or,
$$Q\doteq\tau(1-\delta a_k)+\frac{\mu}{2}\tau\left(\frac{1}{m}-\tau\right)-\frac{\tau\delta}{2}+\frac{1}{12}\delta(\mu+\delta)\tau\left(1-\frac{1}{m^2}+3\frac{\tau}{m}-2\tau^2\right)+\frac{1}{12}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)$$
neglecting the term of the third order.

This, it will be remembered, is the value of $\frac{1}{m}$ payable at the instant of death, if it should occur during the time τ which commences any m-part of a year.

To test this, we see that making $\tau=0$, it vanishes, as it should.

We will further test it by applying it to an annuity certain. Making then $\mu=0$, $\frac{D''_k}{D_k}=\delta^2$, $a_k=\frac{1}{i}$, it becomes

$$\tau\left(1-\frac{\delta}{i}\right)-\frac{\tau\delta}{2}+\frac{\tau\delta^2}{12}=\tau\left(\frac{\delta}{2}-\frac{\delta^2}{12}\right)-\frac{\tau\delta}{2}+\frac{\tau\delta^2}{12}=0.$$

This also is correct; since in this case, there can be no death and therefore no payment.

If in the above value of Q, we suppose $\tau=\frac{1}{m}$, it is clear that we shall obtain the value of $\frac{1}{m}$ payable at the instant of death *whenever it may occur*. Making this substitution, Q becomes $\frac{1-\delta a_k}{m}-\frac{\delta}{2m}+\frac{\delta(\mu+\delta)}{12m}$; whence the value of £1 payable at the instant of death is

$$1-\delta a_k-\frac{\delta}{2}+\frac{\delta(\mu+\delta)}{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

We have thus proved that

$$Q=\tau(1-\delta a_k)-\frac{\tau\delta}{2}+\frac{\mu}{2}\tau\left(\frac{1}{m}-\tau\right)+\frac{1}{12}(\mu+\delta)\delta\tau-\frac{\tau}{12}\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)\left\{\delta(\mu+\delta)-\frac{D''_k}{D_k}\right\}.$$

Also,

$$P=\frac{1-\delta a_k}{2m}-\frac{\delta}{4m}-\frac{\mu}{12m^2}+\frac{(\mu+\delta)\delta}{24m},$$

and from (34)

$$-R=-\tau(1-\delta a_k)+\frac{\tau\delta}{2}-\frac{1}{12}(\mu+\delta)\delta\tau.$$

$$\therefore\ P+Q-R=\frac{1-\delta a_k}{2m}-\frac{\delta}{4m}-\frac{\mu}{12m^2}+\frac{\mu}{2}\tau\left(\frac{1}{m}-\tau\right)+\frac{(\mu+\delta)\delta}{24m}-\frac{\tau}{12}\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)\left\{\delta(\mu+\delta)-\frac{D''_k}{D_k}\right\}.$$

This therefore is the value of the correction. Add now the value of the annuity, $\mathbf{a}^{(m)}_{k\overline{|\tau}}$,

or

$$a_k+\frac{m+1}{2m}-\tau-\frac{\mu+\delta}{12m^2}(m^2-1+6m\tau-6m^2\tau^2)-\frac{1}{12}\frac{D''_k}{D_k}\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)$$

and we get the value of the complete annuity payable m times a year, the first payment being due at the time τ,

$$a_k\left(1-\frac{\delta}{2m}\right)+\frac{1}{m}+\frac{1}{2}-\frac{\delta}{4m}-\tau-\frac{\mu}{12m^2}+\frac{\mu}{2}\tau\left(\frac{1}{m}-\tau\right)+\frac{(\mu+\delta)\delta}{24m}-\frac{(\mu+\delta)}{12m^2}(m^2-1+6m\tau-6m^2\tau^2)-\frac{\tau}{12}\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)\delta(\mu+\delta)$$

$$=\left(a_k+\frac{1}{2}\right)\left(1-\frac{\delta}{2m}\right)+\frac{1}{m}-\tau-\frac{\mu}{12m^2}+\frac{(\mu+\delta)\delta}{24m}-\frac{m^2-1}{12m^2}(\mu+\delta)+\frac{\mu}{2}\tau\left(\frac{1}{m}-\tau\right)-\frac{\mu+\delta}{2}\tau\left(\frac{1}{m}-\tau\right)-\frac{(\mu+\delta)\delta}{12}\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)$$

$$=\left(a_k+\frac{1}{2}\right)\left(1-\frac{\delta}{2m}\right)+\frac{1}{m}-\tau-\frac{\mu+\delta}{12}\left(1-\frac{\delta}{2m}\right)+\frac{\delta}{12m^2}-\frac{\delta}{2}\tau\left(\frac{1}{m}-\tau\right)-\frac{\mu+\delta}{12}\delta\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)$$

$$=\left(a_k+\frac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2m}\right)+\frac{\delta}{12m^2}+\left(\frac{1}{m}-\tau\right)\left\{1-\frac{\delta\tau}{2}-\frac{(\mu+\delta)\delta}{12}\tau\left(\frac{1}{m}-2\tau\right)\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (35)$$

If now we test this result by making $\tau=\frac{1}{m}$, it becomes $\left(a_k+\frac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2m}\right)+\frac{\delta}{12m^2}$, which we see is correct by (23), vol. xiii., p. 377.

Next make $\tau=0$, and we get $\left(a_k+\frac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2m}\right)+\frac{\delta}{12m^2}+\frac{1}{m}$, which is also correct.

If, again, we make m infinite, we have $\tau=0$, since $\tau<\frac{1}{m}$, and the value becomes $a_k+\frac{1}{2}-\frac{\mu+\delta}{12}$.

Next, in the case of a perpetual annuity certain, (35) becomes

$$\left(\frac{1}{i}+\frac{1}{2}-\frac{\delta}{12}\right)\left(1-\frac{\delta}{2m}\right)+\frac{\delta}{12m^2}+\left(\frac{1}{m}-\tau\right)\left\{1-\frac{\delta\tau}{2}-\frac{\delta^2\tau}{12}\left(\frac{1}{m}-2\tau\right)\right\}; \quad \text{since } \mu=0, \text{ and } \frac{D''_k}{D_k}=\delta^2.$$

This ought to agree with (27), vol. xiii., p. 380, when the same substitutions are made, *i. e.* with

$$\frac{1}{i}+\frac{m+1}{2m}-\tau-\frac{\delta}{12m^2}(m^2-1+6m\tau-6m^2\tau^2)-\frac{\delta^2}{12}\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)$$

$$=\frac{1}{i}+\frac{1}{2}+\frac{1}{2m}-\tau-\frac{\delta}{12}+\frac{\delta}{12m^2}-\frac{\delta\tau}{2}\left(\frac{1}{m}-\tau\right)-\frac{\delta^2}{12}\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)$$

The difference between these expressions is

$$\left(\frac{1}{i}+\frac{1}{2}-\frac{\delta}{12}\right)\left(1-\frac{\delta}{2m}\right)+\frac{1}{2m}-\frac{1}{i}-\frac{1}{2}+\frac{\delta}{12}$$

which is easily seen to vanish by means of the value of $\frac{1}{i}$ given by (15), vol. xiii., p. 371.

The value of the perpetual annuity certain in this case is

$$\frac{1}{m}\left\{v^{\tau}+v^{\tau+\frac{1}{m}}+v^{\tau+\frac{2}{m}}+\ldots\ldots\right\}=\frac{1}{m}\cdot\frac{v^{\tau}}{1-v^{\frac{1}{m}}}$$

$$=\frac{1}{\delta}+\frac{1}{2m}-\tau+\delta\left(\frac{1}{12m^2}-\frac{\tau}{2m}+\frac{\tau^2}{2}\right)-\frac{\delta^2}{12}\tau\left(\frac{1}{m}-\tau\right)\left(\frac{1}{m}-2\tau\right)+\frac{\delta^3}{24}\left\{\tau^2\left(\frac{1}{m}-\tau\right)^2-\frac{1}{30m^4}\right\}-\ldots\ldots$$

Lastly, make $\tau=\frac{1}{2m}$; then (35) becomes

$$\left(a_{k}+\frac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2m}\right)+\frac{\delta}{12m^2}+\frac{1}{2m}\left(1-\frac{\delta}{4m}\right)=\left(a_k+\frac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2m}\right)+\frac{1}{2m}-\frac{\delta}{24m^2}$$

and here making $m=1, 2, 4$, successively, we get the correct formulæ for the values of complete annuities,

payable yearly $\left(a_k+\frac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2}\right)+\frac{1}{2}-\frac{\delta}{24}=\left(a_k-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2}\right)+1-\frac{7}{24}\delta$ . . . . (36)

" half-yearly $\left(a_k+\frac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{4}\right)+\frac{1}{4}-\frac{\delta}{96}=\left(a_k-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{4}\right)+\frac{3}{4}-\frac{13}{96}\delta$ . . . . (37)

" quarterly $\left(a_k+\frac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{8}\right)+\frac{1}{8}-\frac{\delta}{384}=\left(a_k-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{8}\right)+\frac{5}{8}-\frac{25}{384}\delta$ . . . (38)

These will be the proper formulæ for calculating accurately the liability at any time of an Insurance Company in respect of the annuities it has granted—these annuities being, in practice, always payable up to the day of death.

The Policies of Assurance Act, 1867.

WE give below a copy of the "Policies of Assurance Act" passed in the last session of Parliament. The Bill, as originally brought in by Sir Colman O'Loghlen, was much shorter, and open to various objections. It was hardly discussed at all in either House of Parliament, but nevertheless underwent many changes during its progress, which was watched with great interest out of doors, several meetings of actuaries and managers of Life Offices having been held in London and Edinburgh to consider the subject.

The original Bill introduced by Sir Colman O'Loghlen passed the House of Commons with little alteration. In the House of Lords it was referred to a Select Committee, which held one meeting only, and whose proceedings, judging from their report, were of the most summary character. No evidence was taken, but the committee cancelled the original Bill altogether, substituting for it an entirely new one, submitted to them by the managers of the Scotch Insurance Companies, and drawn by two counsel, who at the same time gave an opinion altogether hostile to the principle of making Policies assignable at law. The consequence was that the new Bill, as was remarked by Mr. Russell Gurney in the House of Commons, while professing to make policies legally assignable, contained words which were intended to render it practically inoperative. The Bill, thus amended, was sent back to the House of Commons, where several amendments were proposed by Sir Colman O'Loghlen, after consultation with a Committee of Actuaries in London, and carried. These amendments were mostly accepted by the House of Lords; but the very important one, to omit the words in the first clause, which we have placed in italics, was rejected. The result is the Act of Parliament as it now stands.

By its provisions the new principle is now for the first time introduced into our law, that the assignee of a life insurance policy may sue in his own name, and give a legal discharge to the Company for the policy monies, instead of being compelled to sue, and give the discharge, in the name of the original grantee of the policy. But the operation of this principle is greatly limited by the restriction that such assignee must have "the right to give an effectual discharge to the Company"; and doubts are entertained whether mortgagees (as distinguished from purchasers) will in any case be able to sue in their own names.

Furthermore, since it is very rarely that an action is brought to recover the sum assured under a life policy, the importance of the Act to the Insurance Companies is very trifling. They cannot, with any propriety, object to acknowledge notices of assignment, as

required by the Act; and the great majority have long been in the habit of doing so. Those Offices again, which have not hitherto stated on their policies their principal place of business, will find no hardship in doing so for the future.

The Act would have been much more acceptable to the Offices if it had enabled them in all cases to pay the sum assured under a policy to the mortgagee, without requiring the concurrence of the personal representatives of the assured, in the absence of a positive notice from them to the contrary. It is greatly to be hoped that a measure will shortly be carried for securing that object, which would be advantageous both to the Offices and to the public.

A correspondent writes to us on the subject of the Act. "It "is unfortunate, in my opinion, that the operation of the Act is "restricted to assignees 'possessing the right in equity to receive "and the right to give an effectual discharge to the Assurance "Company'—a restriction which it is to be feared will be the "source of much litigation before its precise effect will be under-"stood. It is doubtful, for example, whether a mortgagee of a "policy has the power to give a discharge; and the troublesome "questions which arise on the declaration of a bonus who are the "proper parties to exercise an option as to its appropriation, will "remain. In short, the Offices are in the position of Equity "Judges in the case of every assignment—a function for which "they are not qualified, and which it is neither the interest nor "desire of the public that they should exercise.

"The provisions regarding notice are, I think, judicious. A "notice to an agent will not be binding; on the other hand, some "doubt may arise when the principal place of business of the "Company is removed,—a not uncommon occurrence.

"The Schedule appended to the Act, it is feared, will be a dead "letter, regard being had to the wording of the first clause.

"It is hardly likely that this Act will be final. I believe that "the more the subject is discussed the more the conviction will "grow, that Policies should be as readily transferable as Stock, "Shares, or Bills of Exchange; and that at no distant time this "object will be attained."

ANNO TRICESIMO & TRICESIMO PRIMO VICTORIÆ REGINÆ.

CAP. CXLIV.

An Act to enable Assignees of Policies of Life Assurance to sue thereon in their own Names.

[20*th August*, 1867.]

WHEREAS it is expedient to enable assignees of policies of life assurance to sue thereon in their own names:

Be it enacted by the Queen's most Excellent Majesty, by and with the advice and consent of the Lords Spiritual and Temporal, and Com-

mons, in this present Parliament assembled, and by the authority of the same, as follows:

Assignees of life policies may sue in their own names.

1. Any person or Corporation now being or hereafter becoming entitled, by assignment or other derivative title, to a policy of life assurance, *and possessing at the time of action brought the right in equity to receive and the right to give an effectual discharge to the Assurance Company liable under such policy for monies thereby assured or secured,* shall be at liberty to sue at law in the name of such person or Corporation to recover such monies.

Defence or reply on equitable grounds may be pleaded.

2. In any action on a policy of life assurance, a defence on equitable grounds, or a reply to such defence on similar grounds, may be respectively pleaded and relied upon in the same manner and to the same extent as in any other personal action.

Notice of assignment to be given.

3. No assignment made after the passing of this Act of a policy of life assurance shall confer on the assignee therein named, his executors, administrators, or assigns, any right to sue for the amount of such policy, or the monies assured or secured thereby, until a written notice of the date and purport of such assignment shall have been given to the Assurance Company liable under such policy at their principal place of business for the time being, or in case they have two or more principal places of business, then at some one of such principal places of business, either in England or Scotland or Ireland, and the date on which such notice shall be received shall regulate the priority of all claims under any assignment; and a payment *bonâ fide* made in respect of any policy by any Assurance Company before the date on which such notice shall have been received shall be as valid against the assignee giving such notice as if this Act had not been passed.

Principal places of business to be specified on policies.

4. Every Assurance Company shall, on every policy issued by them after the thirtieth day of September, one thousand eight hundred and sixty-seven, specify their principal place or principal places of business at which notices of assignment may be given in pursuance of this Act.

Assignment by endorsement or separate instrument.

5. Any such assignment may be made either by endorsement on the policy or by a separate instrument in the words or to the effect set forth in the Schedule hereto, such endorsement or separate instrument being duly stamped.

Notices of assignment to be acknowledged.

6. Every Assurance Company to whom notice shall have been duly given of the assignment of any policy under which they are liable shall, upon the request in writing of any person by whom any such notice was given or signed, or of his executors or administrators, and upon payment in each case of a fee not exceeding five shillings, deliver an acknowledgment in writing under the hand of the Manager, Secretary, Treasurer, or other Principal Officer of the Assurance Company of their receipt of such notice; and every such written acknowledgment, if signed by a person being *de jure* or *de facto* the Manager, Secretary, Treasurer, or other Principal Officer of the Assurance Company whose acknowledgment the same purports to be, shall be conclusive evidence as against such Assurance Com-

pany of their having duly received the notice to which such acknowledgment relates.

7. In the construction and for the purposes of this Act the expression "policy of life assurance," or "policy," shall mean any instrument by which the payment of monies, by or out of the funds of an Assurance Company, on the happening of any contingency depending on the duration of human life, is assured or secured; and the expression "Assurance Company" shall mean and include every Corporation, Association, Society, or Company now or hereafter carrying on the business of assuring lives or survivorships, either alone or in conjunction with any other object or objects. **Interpretation of terms.**

8. Provided always, that this Act shall not apply to any policy of assurance granted or to be granted or to any contract for a payment on death entered into or to be entered into in pursuance of the provisions of the Acts sixteenth and seventeenth Victoria, chapter forty-five, and twenty-seventh and twenty-eighth Victoria, chapter forty-three, or either of those Acts, or to any engagement for payment on death by any Friendly Society. **Not to apply to contracts under certain Acts.**

9. For all purposes this Act may be cited as "The Policies of Assurance Act, 1867." **Short title.**

Schedule.

I *A.B.* of, *&c.*, in consideration of, *&c.*, do hereby assign unto *C.D.* of, *&c.*, his executors, administrators, and assigns, the [within] policy of assurance granted, *&c.* [*here describe the policy*]. In witness, *&c.*

HOME AND FOREIGN INTELLIGENCE.

UNDER the above heading, we purpose reprinting, among other matters, the Bonus Reports that are issued from time to time by the Life Insurance Companies of the United Kingdom. We believe that those Reports will be found to have a permanent interest attached to them—exhibiting, as they do, not only the growth of the particular Companies to which they refer, but also to a considerable extent the progress of the practice of life insurance in the country. In a less degree, they will also indicate, when compared together over a long series of years, the gradual change of opinion among actuaries as to the best method of ascertaining and exhibiting the financial position of an Office.

It will be found that the Reports in question differ widely as to the nature and amount of the information they contain; and it would for many reasons be better that there should be a greater degree of uniformity in that respect. Inasmuch, however, as anything in the way of comment on the affairs of particular Companies would be quite out of place in this *Journal*, it is thought better not to draw attention to any instances of redundant or deficient information. The Reports will therefore be given without comment, and, as far as practicable, in the exact form that they are issued by the various Companies. It will be necessary, however, to shorten them, by

omitting such portions as contain no facts relating to the position or progress of the Companies.

It is not intended to make any selection of particular Reports, as it is believed that all the Reports at present published will not be too bulky to find a place in our pages; and if we should overlook the Bonus Report of any Office, we shall be glad to be reminded of the circumstance, and to receive a copy for insertion at the earliest opportunity.—Ed. *J. I. A.*

LAW LIFE ASSURANCE SOCIETY.

Established 1823.

Report of the Directors on the occasion of the Division of the Profits of the Society for the Five Years ending on 31st December 1864.

The 31st day of December 1864 was the date up to which, under the provisions of the Deed of Settlement, an investigation into the Society's Assets and Liabilities was to be made, with the view of ascertaining the amount of Profit which has accrued during the five years which ended on that day, and the distribution of such Profit among the parties severally entitled to participate therein. The Directors have, therefore, caused careful calculations of the Society's Assets and of its Liabilities under all its engagements to be made, and now have the pleasure of meeting the Proprietors and Assured for the purpose of communicating to them the result, and of declaring the amount of Surplus to be divided on this the Sixth Division of the Profits of the Society.

During the five years 1860 to 1864, the Society received in New and Renewal Premiums the sum of £1,487,050, and in Interest upon Investments on the Proprietors' Guarantee Fund £161,708, on the Assurance Fund £833,934, together £995,642, making, as the total Receipts of the Society for the five years, the sum of £2,482,692.

During the same period the amount of Claims upon death has been—

Sums assured by Policies	£1,332,080
Bonuses thereon	504,508
Together . .	£1,836,588

Up to 31st December 1864, the Society had issued 18,056 Policies. Of these 7441 Policies then remained in force, the sums assured thereby being £8,913,569, with Bonuses thereon, outstanding from former Divisions, amounting to £1,747,962.

Proprietors' Guarantee Fund.

Since the 31st December 1859, the date of the last Division of Profits, the Capital amount of this Fund has been . . £735,043 : 15 : 9.

Assurance Fund.

On 31st December 1864—

The value of the Assets of this Fund was . .	£4,405,396 : 4 : 5
The value of the Liabilities was . . .	£3,886,584 : 10 : 5
Leaving as the surplus Profit to be divided in respect of the five years 1860 to 1864 the sum of	£ 518,811 : 14 : 0

Of this surplus one-fifth .... £103,762 : 7 : 0 belongs to the Proprietors.
And the remaining four-fifths £415,049 : 7 : 0 belong to the Assured.

The addition of £103,762 : 7 : 0 to the Proprietors' Guarantee Fund will increase that Fund to £838,806 : 2 : 9.

From the Interest accruing from the Investments of the Proprietors' Guarantee Fund the Directors propose to pay to the Proprietors during the Quinquennial period, 1865 to 1869, an Annual Dividend of £3 : 12 : 0 per Share, free of Income Tax, being a dividend after the rate of 36 per cent per annum on the amount paid up on each Share.

In the Allotment among the Assured of the £415,049 : 7 : 0 falling to them as their share of the Profits now to be divided, the same principles of distribution as were adopted on former occasions have been adhered to. Each existing Assurance which has heretofore had Bonus allotted to it, ranks as an Assurance effected at the date of the last Division of Profits at the then age of the life assured. Each Assurance to which Bonus is now allotted for the first time, ranks according to the period during which it has been in force, and the age of the Life assured at the date when it was effected. A reserve is made in respect of those Policies which do not now participate by reason of their not being of two full years' standing; and should they continue in force to the date of the next Division, they will then participate in respect of the full number of Premiums which shall then have been paid under them.

Of the 7441 Policies in force on 31st December last, 6942, or 93¼ per cent., will share in the Division now to be made.

The amount of Bonus to be added to individual Policies varies with the particulars of each case. In a large number of instances the total Bonus considerably exceeds the amount originally assured by the Policy.

The aggregate amount of the Bonuses now to be added is £594,413.

The amount allotted to each individual Policy will be communicated by letter in the course of a few days.

The Bonuses are added to the Policies in the form of Reversionary sums, payable with the sums assured when the Claims arise. The Reversionary Bonuses may, however, be at any time surrendered to the Society, either in consideration of a cash payment, or of a reduction in the future Annual Premiums payable under the Policy. A printed Table which will accompany each Bonus letter shows the terms upon which such surrender of Bonus may be effected, according to the age next birth-day of the Life Assured.

In very many cases, by the surrender of the Bonus, the future Annual Premiums payable under the Policy may be extinguished, and a considerable Reversionary Bonus still remain attached to the Policy.

EAGLE INSURANCE COMPANY.

Established 1807.

Report of the Directors for the Year and Quinquennium ending 30th June, 1867.

The Report which the Directors have now to make to the Proprietors has reference not only to the progress of the Company during the past year, but also to the more important consideration of its financial condition at the close of another quinquennium, and of the long term of Sixty years from the date of its establishment.

The total Premiums received in the year have been £397,533, and the Interest from Investments, £123,352. The Expenses of Management have amounted to £16,920—this last sum including £3,909 which will, for the most part, now cease.

* * * * *

Adverting now to the results of the quinquennial investigation, which has been in progress for several months, and which has been conducted with all due care and circumspection, it appears, from the Actuary's statement, that there were in force on the 30th June last, the following Policies, viz.:—

16,882 participating, assuring, with additions, £9,305,962, and paying Premiums amounting to £274,517 per Annum;

And 4,096 non-participating, assuring £3,709,743, and paying Premiums amounting to £112,319 per annum.

The total amount Assured in these two classes—viz., £13,015,705—together with some annuities, is found, by the minute and laborious processes used in such investigations, to involve an immediate liability of £6,244,830.

The total Annual Premium receivable—viz., £386,836—is shown, by the like processes, to be now worth £4,506,168, or nearly 12 years' purchase.

The net Liability arising under these large items, viz., £1,738,662, is included in the following statement, which has been verified by the Auditors, and which comprises all the realised and unrealised Assets of the Company, and also all claims against it, immediate or remote.

LIABILITIES.

				£	s.	d.
Interest due to Proprietors				3,533	7	6
Claims on decease of Lives Assured, and additions thereto, unpaid				56,774	14	1
Sundry Accounts				21,947	16	11
Value of Sums Assured				6,244,829	14	6
Proprietors' Fund	177,680	0	0			
Surplus Fund	981,514	13	9			
				1,159,194	13	9
				£7,486,280	6	9

ASSETS.

Amount invested in fixed Mortgages	1,352,448	3	5
Ditto ditto, decreasing Mortgages	194,919	8	2
Ditto ditto, Reversions	585,972	10	7
Ditto ditto, Funded Securities	347,502	11	11
Ditto ditto, Temporary Securities	123,227	13	9
Current Interest on the above Investments	28,987	3	3
Cash and Bills	27,207	17	10
Advanced on Security of the Company's Policies	148,572	3	8
Agents' Balances	28,017	0	3
Sundry Accounts	43,175	3	11
Value of Reassurances	100,082	12	0
Value of Premiums	4,506,167	18	0
	£7,486,280	6	9

Deducting the Sums payable on demand, or at an early maturity, it will be found that the *realised* Assets above set forth amount to

£2,897,856. 10*s.* 3*d.*; and the Proprietors will observe that of this amount £177,680, their paid-up capital, is set apart exclusively for them; £1,738,661. 16*s.* 6*d.* exclusively for the Policyholders, and £981,514. 13*s.* 9*d.* for both. This last item forms the provision for the present Bonus, and the accumulating fund for future Bonuses and Expenses, and it is recommended accordingly that £208,774 be now appropriated for immediate distribution, leaving £772,740. 13*s.* 9*d.* to accumulate, and also to meet the reductions of Premium in respect of the Policies transferred by the National Mercantile Assurance Society, the first of which reductions, it may be remembered, is to take place in 1868.

The share of the present distribution pertaining to the Proprietors will be paid to them, with the dividend, early in October. The portion to be allotted to the Policyholders will be determined as quickly as possible, and notices of the addition made in each case dispatched to them; but this process will necessarily occupy considerable time. Meanwhile, some idea may be formed of the amount of these reversionary additions by an examination of the subjoined Table, which exhibits them in the instance of Assurances twenty years old, effected on lives of various ages at commencement of the risk.

Additions to the Sum of £1,000, assured under Eagle Policies of Twenty Years' standing.

Age at Entry.	Additions prior to 1867.			Additions now made.			TOTAL.		
	£	*s.*	*d.*	£	*s.*	*d.*	£	*s.*	*d.*
23	235	0	0	67	12	0	302	12	0
25	221	5	10	62	16	0	284	1	10
27	213	15	10	59	10	0	273	5	10
31	193	19	7	54	13	0	248	12	7
36	180	6	8	49	14	0	230	0	8
41	167	2	4	46	8	0	213	10	4
46	161	2	2	46	12	0	207	14	2

GUARDIAN FIRE AND LIFE ASSURANCE COMPANY.

Established 1821.

REPORT OF THE ACTUARY ON THE QUINQUENNIAL VALUATION, CHRISTMAS, 1864.

The value of every Policy was first computed separately. The whole of the Policies were then re-classified under the existing ages of the Lives assured, and the total values, at each age ascertained by the Table of Equitable Experience, corrected by the late Mr. GRIFFITH DAVIES from observations made in some other Assurance Companies. This table, which has been in use in this Company for many years, appears to represent more nearly than any other the actual experience of the Company to the present time, and its probable experience hereafter. The rate of interest assumed has been 3 per cent. only, and the whole of the excess of the premiums charged over the net premiums has been reserved for future profits and expenses.

With these explanations the following Balance Sheet shews the position of the Life Assurance Branch at Christmas last.

Balance Sheet, Christmas, 1864.

	£	£
To Value of 4399 Policies Assuring £4,519,123 (less £199,279 re-assured) . . .	2,652,102	
„ Value of £235,127 existing Bonuses, and £526 per annum Life Reductions of premium	171,395	
		2,823,497
„ Balance of Profits, Christmas, 1864: . . .		100,925
		£2,924,422
By Value of Gross Premiums.		
„ £141,507 per annum (less £5840 re-assurance premiums)	1,542,419	
„ Deduct Value of excess of gross over nett Premiums reserved for future profits and contingencies	274,800	
		1,267,619
„ Assets in the Life Branch, Christmas, 1864 . .		1,656,803
		£2,924,422

The deduction from the value of the full premiums for future profits and contingencies comprises not only the full excess of the gross over the net premiums, but also a reserve which prudence requires to be made to meet the great variations in mortality which must be expected from year to year, *so that in no way may the future profits be anticipated, as is frequently the case by other methods of valuation.*

The Balance of Profits thus ascertained at Christmas last amounts to £100,925, available for distribution on the present occasion.

The surplus is smaller than at the previous Division, owing to the claims which fell heavily in the years 1861, 1862, and 1863. The actual number of Claims in the five years only exceeded the estimated number by one, being 759 instead of 758; but the total paid was £926,792 instead of £833,033 expected, the claims being principally amongst the largest and oldest Assurances. In the last year the mortality diminished to the ordinary rate, and in the first five months of the present year the Claims have been £23,265 less than the average of the preceding five years.

Considering the two favorable quinquennial periods which preceded the last it seems to indicate, that, under the varying conditions of Life Assurance and the wide limit of the sums assured by a single policy, five years is scarcely a sufficient period for average results. In complying with the popular wish for more frequent Divisions of profits it must be at the risk of greater fluctuations in the amount to be divided.

Since the investigation in 1859 the Assurances in force have increased by about £180,000.

	No. of Policies.	Sums Assured.	No. of Policies.	Sums Assured.
Assurances in force Christmas 1859			4044	£4,338,858
New Business			1498	1,438,118
			5542	£5,776,976
Claims by Death .	759	£926,792		
Surrendered Policies .	186	171,068		
Lapsed and Expired Policies . . .	198	159,993		
			1,143	1,257,853
Assurances in force Christmas 1864			4,399	£4,519,123

The Proportion of Assurances on the Participating and Non-participating Scale was as follows:—

	No. of Policies.	Sums Assured.
Whole Life	3256	£3,256,223
Joint Lives and other Classes . . .	163	161,596
Total Participating	3419	3,417,819
Whole Life	831	865,545
Joint Lives and other Classes . . .	149	235,759
Total Non-participating	980	1,101,304
Total . .	4399	£4,519,123

The Assets of the Company, in the Life Branch, all invested in first class securities, produce an average interest of £4, 5*s*, 11*d* per cent. and are divided in the following proportions:—

		Capital.
Government Securities		£137,982
Bank Stock		35,522
Mortgages.		930,133
Railway and other Bonds		512,590
Annuities, Reversions, &c.		43,241
Invested Funds		£1,659,468
Claims and Annuities admitted, but not due	£29,791	
Less, Balances due from Agents, Bankers and other Cash Balances. . . .	27,126	
		2,665
		£1,656,803

The income of the Company, from Assurances in force and invested Funds, in the Life Branch, at Christmas last was:—

Premiums	£141,507
Interest and Dividends	71,288
	£212,795

The total Expenses of Management for the last year were £7924.

Since the formation of the Company in 1821, the Life Assurance Policies issued have amounted nearly to £13,000,000.

	No. of Policies	Sums Assured.	No. of Policies.	Sums Assured.
Policies issued from 1821 to Christmas, 1864			. 12,571	£12,938,834
Claims by Death	3647	£4,176,409		
Policies surrendered . . .	1744	2,063,776		
Policies Lapsed, Expired, &c..	2781	2,179,526		
			8172	8,419,711
Assurances in force at Christmas 1864 . .			4399	4,519,123

NORTHERN ASSURANCE COMPANY.

Established 1836.

Quinquennial Life Investigation, as at 31st December, 1865.

In consequence of a disproportionate number of the claims which have emerged during the five years ending that date having fallen upon policies

of large amount, the sum available for distribution amongst the Participating Policyholders (£63,602 10*s.* 6*d.*) is relatively less than that realized during the previous quinquennial period, whilst the Reserve of the Non-participating Fund has been reduced to the extent of £1,892 4*s.* 9*d.* below the amount required to meet the liability in respect of the Policies current in that Branch. In the Annuity Branch also, owing to the mortality having been considerably under the expected rate, the amount standing at the credit of the Fund was less by £297 14*s.* 7*d.* than the present value of the existing Annuities, according to the strict method of valuation adopted in the Actuarial Report. These sums, as will presently be seen, the Directors propose to make good out of the available balance of profit upon the year, so that the Reserves of the two accounts will again stand at their full amounts respectively.

Participation Branch.

Although the available surplus upon this Branch is not so large as might have been anticipated, it is, nevertheless, amply sufficient for the payment of a Bonus which, having regard to the comparatively low rates of premium charged by this Company, must still be considered a high one.

The Bonus which the Directors recommend for addition to the Policies of the Assured under this Branch, out of the profits of the five years ending 31st December, 1865, will be at the former rate of £1 7*s.* 6*d.* per cent. per annum, but payable upon the original sum assured only, instead of upon that amount plus the previous Bonus additions as heretofore, with a further prospective Bonus at the same rate upon all policies which shall become claims before the next Investigation.

In accordance with the intimation made at the last Investigation, Policies only participate in the profits after they have been five years in existence; but the Bonus, when payable, is reckoned from the commencement of the Assurance.

The valuation, it is scarcely necessary to add, has been conducted throughout on the most rigorous principles. For the whole of the Policies of Assurance, the Carlisle Table of Mortality has been employed, and for the Annuities, the English Life Table, No 2. In the Annuity Branch, the rate of interest adopted for the whole has been 3 per cent. The same rate has also been assumed in the Assurance Branch, except in respect of 863 of the older Policies (out of the total existing number of 6503), which were valued at 4 per cent., the rate on which their premiums are based. In the Assurance Branch the value of the entire amount added to the pure premiums in the shape of "loading" for future Bonuses, Expenses, and Contingencies, has, in respect of every policy, been strictly deducted, while, as regards the Annuities, such loadings have been added as are adopted by the Office in the sale of these transactions.

The receipts of 1865 were as follows:—

	Participating.	Non-participating.
Premiums received, less reassurances	£63,948 5 7	£23,452 2 0
Interest on accumulated funds	14,880 4 8	5,065 18 0
	78,828 10 3	28,518 0 0
And the expenses of management	£6,394 16 7	£3,742 10 1

Balance Sheet, as at 31*st December*, 1865.

LIABILITIES.

	£	s.	d.	£	s.	d.
Capital Account	100,000	0	0			
Less Reserved Shares	3,503	0	0			
				96,497	0	0
Reserve Fund (Fire)				100,000	0	0
Accumulated Life Fund—Participation Branch				394,709	3	8
Do. do. Non-Participation Branch				129,065	12	5
Do. do. Annuity Branch				53,025	8	10
Government Fire Insurance Duty collected but not yet payable				5,371	18	10
Shareholders' Dividends unclaimed				1,117	10	9
Outstanding Claims, being Fire Losses in process of adjustment, Life Claims not yet payable, and Annuities not called for				35,118	4	10
Bills payable, being Drafts by distant Agencies not arrived at maturity				3,406	11	3
Balance at the credit of Profit and Loss Account				29,473	15	2
				£847,785	5	9

ASSETS.

	£	s.	d.	£	s.	d.
Moneys Invested				764,091	11	4
On Real Estate	195,184	17	7			
On Leaseholds	5,015	1	8			
On Assignment of Dividends on Stock in the Public Funds, Reversions, &c.	18,112	9	6			
Consols, New and Reduced 3 per Cents.	61,724	12	1			
Birkenhead Improvement Commissioners	19,617	17	4			
Railway and other Debentures	150,671	15	4			
Indian Government Guaranteed Railway Stocks	61,706	15	9			
Colonial (British) Government Bonds	35,486	0	9			
Stocks and Securities of Foreign Governments	34,077	13	2			
On Railway and other Stocks and Shares	114,888	3	9			
On Personal Security, with Assignment of Life Policies	11,393	16	0			
Advances to the Assured on the Security of their Policies, being in all cases within the Surrender value thereof	11,815	3	5			
Company's Premises in Aberdeen, London, Edinburgh, and Melbourne	35,482	13	5			
Sundry other Securities	8,914	11	7			
	£764,091	11	4			
Bills receivable, being remittances not arrived at maturity				4,441	1	6
In the hands of Branch Offices and Agencies				46,825	13	3
In hands of Bankers				25,915	13	6
Interest on Investments accrued but not yet payable				5,681	17	11
Miscellaneous Assets				665	3	9
Cash in hand				164	4	6
				£847,785	5	9

ENGLISH AND SCOTTISH LAW LIFE ASSURANCE ASSOCIATION.

Fourth Division of Profits, 1866.

From the Actuary's Report, and the General Balance-Sheet annexed, it appeared that the Assets and Liabilities of the Office, as at Christmas, 1865, were as follows;—

TOTAL ASSETS, including the Present Value of the NET PREMIUMS	£1,462,578
TOTAL LIABILITIES, including the Present Value of the SUMS ASSURED and previous BONUS ADDITIONS	1,399,360
Leaving a SURPLUS of	£63,218

without encroaching in any way on the Reserve for future Expenses and Profits.

DIVISION OF PROFITS.

IN ACCORDANCE with the PROVISIONS of the DEED of SETTLEMENT, the Surplus was appropriated as follows, viz.:—

1. TO THE ASSURED:—To provide for—

(1.) An Addition to all PARTICIPATING POLICIES, effected for the WHOLE TERM of LIFE and in force at the date of Investigation, of a REVERSIONARY BONUS at the rate of ONE-AND-A-HALF PER CENT. PER ANNUM on the SUM ASSURED, for every FULL ANNUAL PREMIUM paid since the previous Division of Profits.

(2.) An Addition to the ENDOWMENT POLICIES, payable at a given age or at death, entitled to participate in Profits, of a REVERSIONARY BONUS at the rate of ONE PER CENT. PER ANNUM on the SUM ASSURED, for each FULL ANNUAL PREMIUM paid since the previous Division of Profits.

The Present Value of such Reversionary Bonus being . . .	£55,831
2. TO THE PROPRIETORS:—TEN PER CENT. of the Divisible Surplus, so as to INCREASE the DIVIDEND on the Shares for the next FIVE YEARS to FIVE SHILLINGS per Share annually, equivalent to £12. 10*s*. per Cent. per Annum on the sum of £2 per Share originally paid up, or £7. 2*s*. 10*d*. per Cent. per Annum on £3. 10*s*. per Share, the amount, increased by previous Bonus Additions to the Shares, at which they now stand in the Books of the Association	6,321
Making a Total of	£62,152
And leaving a Balance of	1,066
	£63,218

APPROPRIATION OF BONUS.

The BONUS is declared in the first instance as a REVERSIONARY ADDITION to the SUM ASSURED; but it may, if so desired, be exchanged for its EQUIVALENT CASH VALUE, or for a REDUCTION of the ANNUAL PREMIUM either for the next FIVE YEARS or for the REMAINDER OF LIFE. The Policy still remains entitled to future Bonus Additions, which can be appropriated in any one of the ways mentioned.

BONUS ADDITIONS TO POLICIES OF £1,000.

Effected in	Original Sums Assured.	Bonus Additions.	At Christmas, 1865.
1840	£1,000	£425	£1,425
1845	1,000	315	1,315
1850	1,000	240	1,240
1855	1,000	165	1,165
1860	1,000	90	1,090
1865	1,000	15	1,015

THESE BONUSES ARE NOT DEFERRED, BUT VEST IMMEDIATELY ON DECLARATION.

GENERAL BALANCE SHEET.

Dr. LIABILITIES.

	£	s.	d.	£	s.	d.	£	s.	d.
To present Value of all Sums Assured and Past Bonus (£2,517,981).							1,276,321	0	0
„ Capital paid up . . .	£40,000	0	0						
„ Bonus Additions thereto . .	30,000	0	0						
				70,000	0	0			
„ Present Value of Life Annuities (£6,635). .				49,068	0	0			
„ Half-year's Dividend due on Shares . . .				2,226	0	0			
„ Salaries and Sundry Accounts				1,745	0	0			
							123,039	0	0
„ Surplus, viz:—							1,399,360	0	0
To the *Proprietors*, One Tenth of the Profits to be applied by way of Increase to the ordinary Dividend during the next Five Years				6,321	0	0			
To the *Assured*, to provide for Reversionary Bonus Additions to all Policies entitled to participate.				55,831	0	0			
				62,152	0	0			
„ Balance carried forward				1,066	12	3			
							63,218	12	3
							£1,462,578	12	3

Cr. ASSETS.

	£	s.	d.	£	s.	d.
By present Value of future *Net* Premiums (£61,520)				849,883	0	0
„ Assets, as per Auditors' Report, viz.:—						
New 3 per Cent. Annuities (£53,117. 18*s*. 2*d*.)	47,806	6	1			
Cash	6,820	7	10			
Balance due by Agents.	353	1	7			
Houses and Furniture—London, Edinburgh, and Glasgow	20,284	13	4			
Mortgages and other Securities	526,004	3	5			
				601,268	12	3
„ Premiums due and Interest accrued on Investments				11,427	0	0
				£1,462,578	12	3

25*th December*, 1865.

NOTICES OF NEW BOOKS.

Manual of Algebra. By the Rev. Joseph A. Galbraith, M.A., Fellow of Trinity College, and Erasmus Smith's Professor of Natural and Experimental Philosophy in the University of Dublin.

We notice this elementary work in consequence of the author having so far departed from previous usage as to introduce some account of the Theory of Life Annuities and Assurance. The reasons for this course are thus stated in the preface. "In consequence of the magnitude and great importance of the subject, and also because some of its problems involve reasonings and speculations of too refined and abstract a character for the less advanced student, the author has been induced to omit the Doctrine of Chances in its general form and to substitute one of its easiest but most useful applications—the theory of Life Annuities and Assurances so far as it depends on a single life."

As regards the general character of the book, we are glad to see in the Table of Contents several subjects mentioned, such as "Multinomial Theorem (Arbogast's method)," "Calculation of logarithms (Koralek's method)," and "Horner's method of Approximation," which are not usually given in elementary books, but which may very properly find a place in them. We purpose however to confine our remarks to that portion of the work which relates to the Theory of Life Contingencies.

The first remark which suggests itself, is, that if the author's chief object was to give an illustration of the Doctrine of Chances, he would have done better not to limit himself to questions involving one life only, but to give also the elementary theorems as to the values of annuities and assurances on two and three lives, and as to the probabilities of survivorship among them. We are also inclined to think that such details as the use of the D and N columns for the calculation of annuities, which certainly do not illustrate the Doctrine of Chances, would be with greater propriety confined to works specially devoted to the subject of Life Contingencies.

On the above points some difference of opinion will perhaps exist. On another point, however, we believe that there is no room for difference of opinion. We have no doubt that all our readers will agree in our opinion that Mr. Galbraith has acted very unwisely in including in his treatise, and attempting to discuss within a few lines, a number of important questions respecting which practical actuaries are by no means agreed. Thus he has in the course of five pages disposed of the following questions.

1. Management of Life Assurance Companies.
2. The Bonus System.
3. "Valuation of Profits."
4. Advantages and Disadvantages of Bonus.
5. Division of Profits.

Each of these subjects might fairly have claimed a chapter for itself; and they are questions which can scarcely be discussed with advantage by any one who is not practically conversant with the management of Insurance Companies. Accordingly we shall see that our author in his remarks, brief as they are, has fallen into several grave errors.

Our author states that it is the "almost universal" practice of actuaries to employ the Carlisle table of mortality, and 3 per cent. as the rate of interest. The result of our observation is that the Carlisle table is much less frequently used than would appear from the above statement. We believe that the Northampton table of mortality is still used by several actuaries, although very few, if any, are now found openly to defend it, and it has been generally abandoned in favour of tables which make a closer approach to the actual facts. Among these, the *Equitable Experience* of Griffith Davies, the *English Life*, and the *Experience of* 17 *Offices*, have each their supporters. There are also several unpublished tables still in use, although the tendency of the present time is to look upon these with increasing disfavour. Lastly, it is the practice of some actuaries to value by means of an artificial table calculated from the premiums actually charged. Again, with regard to the rate of interest, though it is certainly the most usual course to adopt 3 per cent., still there are several Offices whose liabilities are valued at $3\frac{1}{2}$, or even 4, per cent. On a general review of the question, we are inclined to say that the Carlisle 3 per cent. data, although probably employed more frequently than any other data that could

be specified, are certainly not employed in half the valuations that are made. It is a matter of regret, however, that many Offices in publishing their Bonus Reports do not state upon what data their valuations have been made.

The most important problem which an actuary has to deal with is, the periodical valuation of the liabilities of a Life Office. We notice in passing that instead of the preceding phrase, our author uses the odd one, "Valuation of Profits." He also attaches a novel meaning to the word "Profits," saying, "the Profits are applied in the first instance to defray "the expenses of management"; while it would generally be supposed that those expenses must be defrayed before any profits can be realized. The above problem being so important, it should surely, if considered at all, be treated with the greatest care, and precision. But Mr. Galbraith's treatment of the question shows a strange confusion of ideas. That we may not be thought to misrepresent him, we will quote his own words: "The only way in which the affairs of an assurance office can be "effectually investigated is to find the value of each Policy as in Ex. 7, "p. 510 $\left[\textit{i.e.}\text{ by the formula } 1-\frac{1+a_{x+n}}{1+a_x}\right]$. From the total amount of "assets subtract a sum sufficient to buy up all the policies, *and also a sum* "*sufficient to provide a fund for future management;* any balance which "remains may be considered as fair profit." It will be scarcely necessary to point out that if the policies are valued as Mr. Galbraith directs, the whole of the loading of the premiums is thrown off, and it is quite unnecessary to make any further provision for future expenses of management. Nor need we remind our readers that it is quite as satisfactory to value the policies in classes, as to value them separately, and that in several respects the former method is preferable.

The author then proceeds. "The method usually followed is "equivalent to this, although different in detail. Let P be the sum of the "present values of all the premiums payable to the Office; C the present "value of all the claims that can arise against it; A the total amount of "assets; and M the present value of the expenses of future management; "then

"P + A—C—M=Profit."

We do not know what ground our author may have for supposing this to be "the method usually followed"; but we can assure him that, if the method is ever used, it is in very rare and exceptional instances. Nor is this method equivalent, as he states, to the one previously explained. It would be so, if P denoted the value of the *net* premiums, but P is clearly defined to be the value of the "premiums payable to the Office." This being so, the use of this method would amount to an anticipation of all future profit on existing policies. It is usual to deduct, instead of M, a sum, L, equal to the value of the loading of the premiums, which includes a provision, not only for future expenses, but also for future profits.

Scarcely more satisfactory is the treatment of the subject of "Division of Profits," which we are informed, is "one of the most unsatisfactory "questions connected with the business of life assurance." This may, or may not, be the case; but a treatise on algebra is not the place where we should have expected the information to be given. Our author states, without any limitation, that "the only just mode of distributing a surplus "is to calculate for each policy holder a cash bonus in exact proportion to "the loading on his premium over and above his share of the expense of

Then P'_x = [illegible]
P_x = [illegible]

"management, accumulated at compound interest during the intervals of "division." He is apparently not at all aware that this method can make no claim to accuracy unless the liabilities of the Office have been valued by what are sometimes called the true table of mortality, and the true rate of interest. If, on the contrary, it is thought prudent to make the valuation by a 3 per cent. table, while the Office has really been realizing interest at the rate of, say, $4\frac{1}{2}$ per cent. on all its assets, then the above method would give results very far from equitable. We cannot on the present occasion dwell further on this interesting topic; but hope to recur to it at some future time.

We have next to notice our author's comparison of the "Advantages and Disadvantages of Bonus," which appears to us even more out of character with the nature of the book than anything we have as yet noticed. In making this comparison Mr. Galbraith supposes two persons to insure at the age of 25—the one paying a premium of £2. 6*s*. 6*d*. per £100 for an insurance with profits, and the other insuring without profits and paying £1. 16*s*. $5\frac{1}{2}$*d*. per £100. His conclusion is, that it is far more advantageous to insure without profits than with profits. This conclusion may readily be admitted when the difference between the premiums is as great as is here supposed—amounting to no less than $21\frac{1}{2}$ per cent. of the participating premium; but we cannot for an instant admit that this is a fair statement of the case. In fact no Office, (except under very unusual circumstances,) makes so large a difference as to load its without profit premium 7 per cent. and its with profit premium $36\frac{1}{2}$ per cent.! We trust that in any future edition of the work, all reference to these disputed subjects will be carefully avoided.

With reference to the other parts of the subject, to which the preceding objections do not apply, we regret to say that we notice so many inaccuracies of various kinds that we could not at present recommend the work as a text book for students. We will not dwell upon errors such as theoretical students of a subject are sure to fall into—such, for instance, as the statement that money is never lent at compound interest. But we think that we may reasonably expect a theoretical writer, who is explaining the meaning of the "expectation of life," to be more careful than to say that, taking 1,000 persons of 32 years of age, "it is morally certain that if "their ages at death be added together, the sum will be 33,000, or there-"abouts"—instead of 65,000. Again, can anything be more vague and unsatisfactory—or indeed, more untrue—than the statement that "within "certain limits each living person possesses a tenure of life longer or shorter "according to his age"? We will not dwell on minor inaccuracies, such as speaking of an "annuity of £1 a year,"—of the "annuitant" living to receive the payment of an annuity instead of the "nominee" living—of the "number who live and die throughout the year"; but we think it is not tolerable to speak of a "reversion of £1" when meaning a "reversionary annuity of £1." Nor do we think that standard writers speak of "the first and last annuity" when they wish to denote "the first and the last payments of the annuity." If we seem to be severe in pointing out these blemishes, we would remind our readers that one of the chief benefits of a mathematical course is the precision of thought it requires of the student; and that accuracy of language is the unfailing index, if not the invariable accompaniment, of precision of thought.

The only remaining point we have to notice is that Mr. Galbraith adopts the innovation of defining N_x as equal to $D_x+D_{x+1}+\ldots.$ and yet speaks of this as "Davies's notation." This is introducing a fresh element of confusion into a question already too much involved. According to Davies's notation and formulæ

$$N_x=D_{x+1}+D_{x+2}+\ldots.$$

and the use of Davies's symbol, N_x, to denote a different quantity, is a practice much to be deprecated, as very likely to lead to great confusion and error. All that is necessary to prevent mistakes is the use of a new symbol instead of Davies's; and we would suggest $\mathbb{N}_x$ for the purpose; so that we should have

$$N_x=D_{x+1}+D_{x+2}+\ldots.$$

$$\mathbb{N}_x=D_x+D_{x+1}+D_{x+2}+\ldots.$$

and $$\mathbb{N}_x=N_x+D_x.$$

Then, a_x denoting as usual an annuity on a life, x, if $\mathbf{a}_x$ denote an annuity-due, or an annuity payable in advance, on the same life, we shall have

$$a_x=\frac{N_x}{D_x},\quad \mathbf{a}_x=1+a_x=\frac{\mathbb{N}_x}{D_x}.$$

CORRESPONDENCE.

AN ASSURANCE FALLACY.

To the Editor of the Assurance Magazine.

SIR,—The following problem presents several points of interest.

An assurance of A pounds is to be effected on (x), at an annual premium (ϖ), subject to the condition that interest on the premiums paid up to and including the year of death is to be allowed by the Office, at the rate involved in the tables employed, which rate it is assumed is that realized by the Office. Required ϖ.

Attempt a solution thus:—Since all the interest realized is to be handed over to (x) or his representatives, the Office has obviously nothing but the bare premiums out of which to pay the sum assured. It is, therefore, as regards the Office, the same thing as if no interest were made; and we consequently need take account only of the average number of premiums that will be received from each policyholder. This number being $1+e'_x$ (where e'_x is the *curtate* mean duration of lives aged x), we have

$$\varpi(1+e'_x)=A;$$

whence $$\varpi=\frac{A}{1+e'_x}\quad\ldots\ldots\ldots\quad(1).$$

This is a very singular result. It is independent of the rate of interest; and yet it is obvious that the higher the rate realized by the Office the

greater will be the annual return to (x), and consequently the less the cost of the assurance to him. The foregoing equation therefore cannot be true, and the process by which it is attained must be fallacious.*

But where, then, is the fallacy? It is in the assumption, tacitly made in the so-called solution, that the interest realized by the Office and that payable to the policyholders are identical. They are so, however, only as to *rate*, but not as to *amount*, except during the first year. At the end of that period the premium fund is so reduced by payment of death claims, that the interest yielded by it is no longer sufficient to meet that due to the policyholders. The deficiency, therefore, must be made good from the premiums themselves, and these therefore require to be increased to meet this charge.

The reasons why I have commenced with an erroneous solution, are—first, that an impression prevails, as I am informed, that this solution is a correct one; and secondly, that the problem belongs to a class which appear to invite the application of what are called common sense notions, while such applications usually lead, as in the present case, unless skilfully managed, to erroneous conclusions.

I now give a legitimate solution of the problem. The benefit consists of, first, a uniform assurance of A, the term corresponding to which is AM_x; and secondly, of an increasing annuity of ϖi, $2\varpi i$, $3\varpi i$, &c., which makes its last payment at the end of the year of death. The term given by this annuity, *minus* its last payment, is $\varpi i S_x$, and that given by the last payment is $\varpi i R_x$. Hence, the payment term being ϖN_{x-1}, we have

$$\varpi N_{x-1} = AM_x + \varpi i(S_x + R_x).$$

From this we obtain

$$\varpi = \frac{AM_x}{N_{x-1} - i(S_x + R_x)}.$$

Now,
$$N_{x-1} - i(S_x + R_x) = N_{x-1} - i(S_x + vS_{x-1} - S_x)$$
$$= N_{x-1} - (1-v)S_{x-1} = R_x.$$

$$\therefore\ \varpi = \frac{AM_x}{R_x} \quad . \ . \ . \ . \ . \ . \ . \quad (2).$$

Of the value of ϖ thus determined it would be easy to show that for any value of x, except the oldest age in the table (for which ϖ is always equal to A), it increases *with* (not *as*) i, the rate of interest.

Since, when i diminishes without limit, $\frac{M_x}{R_x}$ approaches without limit to $\frac{D_x}{N_{x-1}}$; therefore, when $i=0$, *i.e.*, when money bears no interest, we have

$$\varpi = \frac{AD_x}{N_{x-1}} = \frac{A}{1+\acute{e}_x},$$

which agrees with (1). From this it appears that, although not true generally, (1) is true in the case of money bearing no interest. In this

* The reasoning here does not seem quite conclusive.—Ed. *J. I. A.*

case, however, no interest being realized there is none payable to the policyholders.

The following table shows the premium per cent., by the Carlisle rate of mortality, at several rates of interest. The commutation table for $i=0$ will be found at p. 145, vol. xiii. of the *Journal of the Institute of Actuaries.*

Age.	$i=0.$	$i=\cdot03.$	$i=\cdot04.$	$i=\cdot05.$
30	2·8604	3·7290	4·1146	4·5707
50	4·6281	5·4515	5·7719	6·1156
70	10·3371	11·6040	12·0426	12·4872
90	26·4432	28·5903	29·3248	29·9864

For further elucidation of this somewhat curious problem I have worked out the following example at length, by the Carlisle table, at 5 per cent. The age is 90, and the sum assured £100. By (2) we get for the annual premium

$$\varpi=\frac{147\cdot9288}{4\cdot933192}=29\cdot98643;\text{ whence }\varpi i=1\cdot4993215.$$

142ϖ		4258·0731			1732·4009
5 per cent.		*212·9037			*86·6200
		4470·9768			1819·0209
$\varpi i\times 142$	*212·9037		$5\varpi i\times 40$	*299·8643	
100×37	3700·	3912·9037	100×10	1000·	1299·8643
		558·0731			519·1566
105ϖ		3148·5752	30ϖ		899·5929
		3706·6483			1418·7495
		*185·3324			*70·9375
		3891·9807			1489·6870
$2\varpi i\times 105$	*314·8575		$6\varpi i\times 30$	*269·8779	
100×30	3000·	3314·8575	100×7	700·	969·8779
		577·1232			519·8091
75ϖ		2248·9822	23ϖ		689·6879
		2826·1054			1209·4970
		*141·3053			*60·4794
		2967·4107			1269·9719
$3\varpi i\times 75$	*337·3473		$7\varpi i\times 23$	*241·3908	
100×21	2100·	2437·3473	100×5	500·	741·3908
		530·0634			528·5811
54ϖ		1619·2672	18ϖ		539·7557
		2149·3306			1068·3368
		*107·4665			*53·4168
		2256·7971			1121·7536
$4\varpi i\times 54$	*323·8534		$8\varpi i\times 18$	*215·9023	
100×14	1400·	1723·8534	100×4	400·	615·9023
		532·9437			505·8513
40ϖ		1199·4572	14ϖ		419·8100
		1732·4009			925·6613

		925·6613
		*46·2830
		971·9443
$9\varpi i \times 14$	*188·9143	
100×3	300·	488·9145
		483·0308
11ϖ		329·8507
		812·8815
		*40·6440
		853·5255
$10\varpi i \times 11$	*164·9254	
100×2	200·	364·9254
		488·6001
9ϖ		269·8779
		758·4780
		*37·9239
		796·4019
$11\varpi i \times 9$	*148·4328	
100×2	200·	348·4328
		447·9691
7ϖ		209·9050
		657·8741
		*32·8937
		690·7678
$12\varpi i \times 7$	*125·9430	
100×2	200·	325·9430
		364·8248
5ϖ		149·9322
		514·7570

		514·7570
		*25·7378
		540·4948
$13\varpi i \times 5$	*97·4559	
100×2	200·	297·4559
		243·0389
3ϖ		89·9593
		332·9982
		*16·6499
		349·6481
$14\varpi i \times 3$	*62·9715	
100×2	200·	262·9715
		86·6766
ϖ		29·9864
		116·6630
		*5·8331
		122·4961
$15\varpi i \times 1$	*22·4898	
100×1	100·	122·4898

Little explanation of the above is needed. At the outset the premium is received from the tabular number alive at 90, viz., 142, and a year's interest is added, giving a total in hand at the end of the first year of £4,470. This is immediately reduced by the payment of, first, £212·9037, interest on the premiums, and secondly, £3,700, the claims arising on 37 deaths, to £558·0731. The premium is again received from the 105 survivors, a year's interest is added, and the outgoings of the second year, amounting to £3314·8575, are deducted, leaving £577·1232 in hand at the commencement of the third year. In this way the scheme works itself out at the end of the fifteenth year.

It is visible now that after the first year the interest which the office realizes is altogether insufficient to meet that which it has to pay. And it is singular to note that, after the first few years, the ratio of the interest receivable (by the Office) to the interest payable, closely approximates to that of 1 : 4.* Whether this is accidental, or whether the like would be observed in other circumstances, I am at present unable to say.

Returning to equation (2), and writing it thus,

$$\varpi R_x = AM_x,$$

* To facilitate this comparison I have marked the interest on both sides with asterisks.

we see that the transaction resolves itself into an exchange or commutation of one assurance on (x) for another, viz., a uniform assurance of A payable *by* the Office, and an increasing assurance of ϖ, 2ϖ, &c. ($n\varpi$ in the nth year), payable *to* the Office. And this is correct, as it is obviously the same thing, theoretically, whether the premiums be paid annually, interest being allowed upon them, or in the aggregate at the end of the year of death. In practice, however, there is a great distinction between the two modes of payment. No Office would consent to defer the receipt of premium till the emergence of the claim, as they would in a great many cases have then more to receive than to pay.

It is interesting, however, to watch the operation of this mode of payment in a particular case; and I have therefore worked it out for the same age as before, 90, and at the same rate, 5 per cent. The premium also is of course the same, 29·98643.

100×37	3700·				
$\varpi \times 37$	1109·4979	2590·5021			2852·6285
5 per cent.		129·5250			142·6314
		2720·0271			2995·2599
100×30	3000·		100×3	300·	
$2\varpi \times 30$	1799·1858	1200·8142	$9\varpi \times 3$	809·6336	−509·6336
		3920·8413			2485·6263
		196·0420			124·2813
		4116·8833			2609·9076
100×21	2100·		100×2	200·	
$3\varpi \times 21$	1889·1451	210·8549	$10\varpi \times 2$	599·7286	−399·7286
		4327·7382			2210·1790
		216·3869			110·5090
		4544·1251			2320·6880
100×14	1400·		100×2	200·	
$4\varpi \times 14$	1679·2401	−279·2401	$11\varpi \times 2$	659·7015	−459·7015
		4264·8850			1860·9865
		213·2442			93·0493
		4478·1292			1954·0358
100×10	1000·		100×2	200·	
$5\varpi \times 10$	1499·3215	−499·3215	$12\varpi \times 2$	719·6743	−519·6743
		3978·8077			1434·3615
		198·9404			71·7181
		4177·7481			1506·0796
100×7	700·		100×2	200·	
$6\varpi \times 7$	1259·4301	−559·4301	$13\varpi \times 2$	779·6472	−579·6472
		3618·3180			926·4324
		180·9159			46·3216
		3799·2339			972·7540
100×5	500·		100×2	200·	
$7\varpi \times 5$	1049·5250	−549·5250	$14\varpi \times 2$	839·6200	−639·6200
		3249·7089			333·1340
		162·4854			16·6567
		3412·1943			349·7907
100×4	400·		100×1	100·	
$8\varpi \times 4$	959·5658	−559·5658	$15\varpi \times 1$	449·7965	−349·7965
		2852·6285			

The great distinction between this mode of arranging the transaction and the other is that there the Office was put in funds at the outset, enabling it to meet all claims as they arose, while here it is in advance from first to last.

If it is required to *load* the premium of this problem, we must proceed as in all cases in which the Office makes a return to the assured. It is not sufficient to apply the required loading to the value of ϖ, determined as above, since this would leave the additional interest which has to be returned unprovided for. The loading must, as in all such cases, be applied to the benefit side of the fundamental equation.

Let the required loading be k per pound. Then,

$$\varpi N_{x-1} = (1+k)\{AM_x + \varpi i(S_x+R_x)\};$$

whence,
$$\varpi = \frac{(1+k)AM_x}{N_{x-1}-(1+k)i(S_x+R_x)}.$$

But,
$$N_{x-1}-(1+k)i(S_x+R_x) = N_{x-1}-(1+k)i(S_x+vS_{x-1}-S_x)$$
$$= N_{x-1}-(1+k)(1-v)S_{x-1} = (1+k)\{N_{x-1}-(1-v)S_{x-1}\}-kN_{x-1}$$
$$= (1+k)R_x - kN_{x-1}.$$

$$\therefore\ \varpi = \frac{(1+k)AM_x}{(1+k)R_x-kN_{x-1}} = \frac{AM_x}{R_x-\frac{k}{1+k}N_{x-1}} \quad \ldots\ldots (3).$$

This is obviously greater than $\frac{AM_x}{R_x}$; but it can be shown to be also greater than $\frac{(1+k)AM_x}{R_x}$, which is what the net premium becomes when the loading is *directly* applied to it. Thus,

$$\frac{AM_x}{R_x-\frac{k}{1+k}N_{x-1}} > (1+k)\frac{AM_x}{R_x},$$

if $\qquad R_x > (1+k)R_x - kN_{x-1},$

if $\qquad kN_{x-1} > kR_x,$

if $\qquad N_{x-1} > R_x;$

and this last we know to be true.

If no interest is earned, M_x and R_x, as before, assume their limiting values, and (3) becomes

$$\varpi = \frac{AD_x}{N_{x-1}-\frac{k}{1+k}N_{x-1}} = (1+k)\frac{AD_x}{N_{x-1}}.$$

In this case, therefore, it suffices to apply the loading *directly* to the net premium; which is in accordance with the remark already made, the interest returnable by the Office being here *nil*.

I append a table of loaded premiums, corresponding to that already given of net premiums. The loading is 10 per cent., that is $k = \cdot 1$.

Age.	$i=0$.	$i=\cdot03$.	$i=\cdot04$.	$i=\cdot05$.
30	3·1464	4·5002	5·1881	6·1767
50	5·0909	6·3154	6·8386	7·4329
70	11·3708	13·0714	13·6844	14·3199
90	29·0875	31·8114	32·7722	33·6255

I am, Sir,
Your most obedient servant,

P. GRAY.

London, 2nd Sept., 1867.

*** A short note on the problem which forms the subject of this letter will be found in vol. v., p. 348.

VALUE OF A POLICY—FORMULÆ—MILNE.

To the Editor of the Assurance Magazine.

DEAR SIR,—There is a theorem which I suppose must be in the heads of many actuaries, but I cannot find it in any of the books. It is that the values of a policy, as it runs on, are proportional to the falls in the value of the annuity. That is, if a_x be the value of an annuity of £1 at the age x, the age of creation of the policy, the values of the policy at the ages y and z are as a_x-a_y to a_x-a_z. That this theorem is not commonly expressed seems due to the value at the age y being usually written $1-\frac{1+a_y}{1+a_x}$ instead of $\frac{a_x-a_y}{1+a_x}$.

I shall be curious to see whether any one will produce a statement of this simple form. I find it occasionally very useful to take out from the table, without any writing, that the policy-value of $1+a_x$ at death is a_x-a_y at the age y, the age x being that of commencement. When a formula represents two different results, it is a useful exercise of ingenuity to deduce one result directly from the other. Now a_x-a_y is the value to (x) of a counter-survivorship—as we may call it—of the following kind. The executors of the first who dies pay an annuity of £1 to the survivor; and $(a_x-a_y)\div(1+a_x)$ is the whole-life premium which (x) should pay to be put in this position. How, from the nature of this contract, does it follow that one payment of this premium, over and above the annual premium which (x) should pay, admits (y) to a policy of £1 at the premium for the age (x)?

Easy forms, corollaries from common forms, are things for *second editions*. A person who is engaged in a great effort, and has a heavy system of tables to look after, does not watch offshoots. Now none of the best known works—except only those of Price and Morgan, which lay no stress on formulæ—have arrived at second editions: this may be said of Baily, G. Davies, Milne, and David Jones.

It is much to be regretted that Milne did not, in his later years, occupy himself with a reconstruction of the algebraical part of his work. But it is hardly known how completely he abandoned the subject. In May,

1839, he wrote to me as follows:—"I am far from taking an interest now in investigations of the values of life contingencies; I have long since had too much of that, and been desirous of prosecuting inquiries into the phenomena of nature, which I have always regarded with intense interest."

Long before the above date, Milne had gained an unusually minute knowledge of natural history. When my colleague—as he then was—Macculloch, was putting together his dictionary of political economy, he was puzzled to know the character of some animal whose skin formed an article of commerce imported, I think, from Spain. He applied to zoologists without result: he brought away the impression that they did not take any interest in animals useful to man; and very sarcastic he was—and an unequivocal Scotch tongue is a very effective instrument of sarcasm—upon their imputed feeling in this respect. He knew that his friend Milne had paid attention to natural history, and applied to him in hope of reference to some source of knowledge. Milne immediately gave him all he wanted about the animal, and a great deal more, without book and with perfect precision.

The second quarter of our century was distinguished by the growth in England—and abroad also—of attention to especial points of scientific history, with complete research, and publication of documents, or at least of full reference. Four men are conspicuous; Stephen Peter Rigaud, of Oxford; George Peacock, of Cambridge; Francis Baily, Actuary, and of the Stock Exchange; and Joshua Milne, Actuary of the Sun Life Office. To these might be added John Drinkwater-Bethune, whose scientific biographies are special researches, though full materials were not published. Of the four men first named, two were academics, with large public libraries at their command; accordingly they left but few books. The two commercial men had to collect their own libraries; and they left two very remarkable sale catalogues on their own *amateur* subjects. Baily's library was astronomical, and not rich in life contingency. Nor was Milne's: I suspect he had parted with nearly all that was curious and that especially helped him in his historical articles. But in this collection of more than two thousand lots, representing perhaps six thousand volumes, we find a powerful force of general mathematics, Newton, Euler, D'Alembert, Lagrange, &c., and a very large collection of natural history, medicine, music, &c. I picked up two books connected with music, of which I never heard of other copies for sale: Solomon de Caus, and the collection of Meibomius. At the same time, it looks odd that in the library of the historian of life contingencies *Kerseboom* should be missing, and the contemporary anatomist *Cassebohm* should be present in several works.

There is a very good word coming into use to express the full treatment of one separated point; such a thing is a *monograph.*

The branches of science are becoming so extensive that histories will not be written again for a long time. But monographies will, I hope, abound; and the time may come when there shall be so many of them that somebody may abbreviate the total into a history, referring to the monographers for further detail and for evidence, and laying all responsibility on their shoulders.

Yours truly,

A. DE MORGAN.

A PRACTICAL QUESTION.

To the Editor of the Assurance Magazine.

SIR,—It having been suggested in a letter from Mr. Tucker, in vol. v., p. 255, that practical questions might fitly be commented on in the pages of the Magazine, and Mr. Gray, in a recent volume, having advocated the importance of keeping up the "Correspondence Department," I venture to trouble you with the following case which has occurred in practice, hoping that although it presents no algebraical difficulty in the handling of it, it may yet prove of sufficient interest to attract some of your readers.

"A lady, aged 67 last birthday, holding a jointure well secured, wishes "an advance of £1,000 to enable her to buy a house. What annuity "will an Assurance Company require, the house reverting to the Company "at her death?"

First method.

This may be viewed in the first instance as two separate transactions; and we shall proceed in the first place to determine the annuity, without taking into account the value of the reversion.

Thus, fixing the interest on our advance at 5 per cent., we have the following calculation:—

Premium at age 68=(say)	£9 15 1
	9·754
d=one year's interest at 5 per cent. discounted=	4·762
Annuity *due*, in which the Company must be secured=	14·516

$$\frac{100}{14\cdot516} = \begin{array}{r} 6\cdot889 \\ -1\cdot \\ \hline 5\cdot889 \end{array}$$

5·889=value of annuity of £1, first payment at end of one year.

Then by the proportion 5·889 : 1000 :: 1 : 169·81 we find the annuity required for the advance of £1,000 absolutely to be (say) £170.

In the second place, we have now to determine what deduction should be made from this annuity in consideration of the reversion of the house; and let us consider this reversion equivalent to that of an absolute sum of £750, one fourth being deducted for probable depreciation of the property.

The value of £1 to be received on the death of a person aged 68 next birthday, Carlisle 3 per cent., is . . . . . . ·74168

And multiplying by	$7\frac{1}{2}$
	37084
	519176
We have value of reversion of £750=	556·260

Now the annuity which by the Office Tables £100 will purchase is (say) £10. 13*s*.: therefore 10·65×5·5626=59·241 gives the annuity (payable

yearly) to be granted for purchase price of £556. 5*s*. 2*d*., and deducting it from the annuity required by the Company, as above £170 0 0

from the annuity required by the Company, as above	£170	0	0
	59	4	10
We have the net annuity required . . .	£110	15	2

Second method.

The transaction, however, may be looked upon in another light.

If we look upon the reversion as equivalent to that of an absolute sum of £750, all that the Company require to assure on the life 68, is the amount of depreciation, £250.

That being the case, the calculation will stand thus:—

$$\begin{aligned} \text{One-fourth of } p &= 2{\cdot}438 \\ d &= 4{\cdot}762 \end{aligned}$$

Annuity *due* in which the Company must be secured $7{\cdot}200$

$$\frac{100}{7{\cdot}2} = 13{\cdot}889$$
$$\underline{-1}$$
$12{\cdot}889$ = value of annuity of £1, first payment at end of one year.

Then by the proportion $12{\cdot}889 : 1000 :: 1 : 77{\cdot}585$, we have £77. 11*s*. 8*d*. as the net annuity which the Company require to secure them, a seemingly fair rate.

If we adopt the second view of the case, the transaction partakes more of the nature of an advance on security, and involves the consideration of the desirableness of lending on house property. But taking into account the extent to which depreciation is provided against, I think the security may be held to be good.

The first method is that which would be adopted, in each case, if the two proposals contained in the transaction were made by different persons.

Between the limits there is a wide range for fixing the rate, and I shall be glad to have the opinion of any gentleman as to what may be thought an equitable one.

I am, Sir,

Your most obedient servant,

Edinburgh, 19*th September*, 1867. J. C.

P.S.—Under the first method, the value of the reversion has been taken so as to bring out the most favourable value for the proposer. If it had been found by the usual formula $\frac{1-ia_x}{1+i}$,—taking a_x from the Office tables and i at 5 per cent. from Orchard*—it would have been only £379. 1*s*. 3*d*., which would have brought out the difference between the rates of annuity required under the two methods, still greater.

* Is our correspondent correct in terming this the "usual formula"? As regards the problem, we should ourselves be disposed to adopt his *Second method*, taking *d* at *six* per cent., $= {\cdot}05660$, which would give £88. 2*s*. 5*d*. as the annuity to be received by the Company.—Ed. *J. I. A.*

JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

Briggs's Method of Interpolation; being a translation* of the 13th Chapter and part of the 12th of the Preface to the "Arithmetica Logarithmica," by* J. Hill Williams, *Esq., one of the Vice-Presidents of the Institute of Actuaries.

[Read before the Institute, 30th December, 1867.]

THOSE of our readers who have studied the paper of M. Maurice on Interpolation, of which a translation appeared in our last number, will no doubt be glad to compare with it Briggs's own description of his method of Interpolation. His original work however appears to be very scarce; and the chapters in which he describes his method—the 12th and 13th—are omitted, even in the Edition published by Vlacq in Briggs's lifetime. We believe, therefore, that this translation by Mr. Williams of those parts of Briggs's Preface in which he describes his method of Interpolation, will prove very acceptable to our readers.—Ed. *J. I. A.*

Chapter XII.

Given two consecutive integers and their Logarithms: it is required to interpolate between them nine other equidistant numbers, and to find their Logarithms.

If the second differences of the given Logarithms are nearly equal, this will be an easy matter: but if the third differences cannot be neglected, this method will be found somewhat defective.

* Briggs, like most, if not all, of his contemporaries, wrote in Latin.

Take two consecutive numbers A, and their Logarithms B, together with their first differences C, and their second differences D. If the second differences are equal, multiply either of them into the numbers standing opposite the first ten natural numbers in the subjoined Table E; then, the three last figures having been cut off each of the products F, G, H, I, K, the first five are to be added to the tenth part of the first difference of the two given logarithms, and the last five are to be subtracted from the same. The sums and the remainders will be the differences of the Logarithms sought; and the successive addition of these differences to the smaller of the given Logarithms, will give the Logarithms required. For example, let the given numbers be 91235 and 91236, the first difference of their logarithms being 47601,4799.

	47602,0016.C	
91235.A.	4·96016,14763,8639.B	5217.D
	47601,4799.C	
91236.A.	4·96016,62365,3438.B	5217.D
	47600,9582.C	

TABLE E.

1	45	Products to be added.
2	35	
3	25	
4	15	
5	5	
6	5	Products to be subtracted.
7	15	
8	25	
9	35	
10	45	

Natural numbers.	Logarithms.		
912350	4·96016,14763,8639		
		4760,1715	C+F
1	4·96016,19524,0354		
		4760,1662	C+G
2	4·96016,24284,2016		
		4760,1610	C+H
3	4·96016,29044,3626		
		4760,1558	C+I
4	4·96016,33804,5184		
		4760,1506	C+K
912355	4·96016,38564,6690		
		4760,1454	C−K
6	4·96016,43324,8144		
		4760,1402	C−I
7	4·96016,48084,9546		
		4760,1350	C−H
8	4·96016,52845,0896		
		4760,1297	C−G
9	4·96016,57605,2193		
		4760,1245	C−F
912360	4·96016,62365,3438		

Products.		5217. Multiplicand.	
F ..234	765	45	Multipliers.
G ..182	595	35	
H ..130	425	25	
I .. 78	255	15	
K .. 26	085	5	
47601479	9	$\frac{1}{10}$C	
47601714	7.	C+F	
47601662	5.	C+G	
47601610	3.	C+H	
47601558	2.	C+I	
47601506	0.	C+K	
47601453	8.	C−K	
47601401	6.	C−I	
47601349	5.	C−H	
47601297	3.	C−G	
47601245	1.	C−F	

If the second differences are unequal, as below:* add the two consecutive second differences, take half the sum for the second difference, and multiply as before.

	4,51707,8187	C	
9615.A	3·98294,92885,7450	B	469771 D*
	4,51660,8416	C	
9616.A	3·98299,44546,5866	B	469672 D
	4,51613,8744	C	
			939443 Sum
			469721 ½ Sum

Products.		469721 Multiplicand
F. 21137	445	45 Multipliers
G. 16440	235	35
H. 11743	025	25
I. . 7045	815	15
K. . 2348	605	5

96150	3·98294,92885,7450	¶
	45168,1979	C+F
1	3·98295,38053,9429	
	45167,7282	C+G
2	3·98295,83221,6711	
	45167,2585	C+H
3	3·98296,28388,9296	
	45166,7887	C+I
4	3·98296,73555,7183	
	45166,3190	C+K
96155	3·98297,18722,0373	
	45165,8493	C−K
6	3·98297,63887,8866	
	45165,3796	C−I
7	3·98298,09053,2662	**
	45164,9099	C−H†
8	3·98298,54218,1761	
	45164,4401	C−G
9	3·98298,99382,6162	
	45163,9704	C−F
96160	3·98299,44546,5866	

451660841	6	$\frac{1}{10}$C
451681979	0	C+F
77281	8	C+G
72584	6	C+H
67887	4	C+I
63190	2	C+K
451658493	0	C−K
53795	8	C−I
49098	6	C−H
44401	4	C−G
39704	2	C−F

469721
105
2348605
469721
49320|705 §

4516608416
7
Products { 3161625891|2
49320|7 §
3161675212
¶ 398294928857450
** 96157 398298090532662
451660841|6 $\frac{1}{10}$C
Product. . 11743|0
† Remainder 4516490986|6

Table E′.

	Products to be added
1	45
2	80
3	105
4	120
5	125
6	120
7	105
8	80
9	45

But, suppose you wish to find any one of the logarithms without the others. Multiply the number less than ten which is written at the end of the given number A, into the given difference C; multiply also the number standing opposite to it in Table E′ into the second difference; and, cutting off three figures from the latter

product, and one from the former, add the products: the sum added to the given Logarithm will give the Logarithm sought. If, for example, you wish to know what is the Logarithm of the number 96157, the process is as follows. The given difference 4516608416 is to be multiplied by 7. The product is 31616258912. Then taking 105, the number opposite to 7 in Table E′, multiply it into the second difference 469721; add the product 49320 | 705 (with the three last figures cut off) to the first product (with one figure cut off) 3161625891. Add the total 3161675212 to the given Logarithm ¶, and the total, 398298090532662, will be the required Logarithm of the number 96157. If you wish to know the difference between the Logarithm of this number and that of the next higher number: multiply the number in the Table standing opposite the number 8, which is greater by unity than the given 7, into the second difference 469721; subtract the product 11743|025 (with three figures cut off) from the tenth part of the given first difference, and the remainder 451649099 will be the difference sought.†

[The remainder of this Chapter describes the method of finding the number corresponding to a given Logarithm. It would not be intelligible without quotations from former Chapters; and as it does not illustrate the direct method of interpolation, it is here omitted.]

Chapter XIII.

To find the Logarithms of the omitted Thousands of natural numbers [20,000 to 90,000, not calculated in his Tables]; or, given any equidistant numbers whatsoever, together with their Logarithms, to find the Logarithms of the four numbers interpolated at equal intervals between each adjacent two.

The intermediate Logarithms may be obtained in various ways. I think the following is the best way; the others we will consider afterwards.

Take the first, second, third, fourth and other differences of the given Logarithms; and divide the first differences by 5, the second by 25, the third by 125, and so on; the divisors increasing in a quintuple ratio; and call the quotients the first, second, third, &c., *mean* differences. Or, instead of dividing, multiply the first given differences by 2, the second by 4, the third by 8, and so on; cutting off in the products, one figure from the first product, two from the next, three from the third, and so on: [*i.e.* multiply the

given differences respectively by ·2, by ·04, by ·008, &c.] These products (which are equal to the quotients above-described) will be the first, second, third, &c., mean differences. For example, let the following Logarithms be given, together with their first, second, third, fourth and fifth differences, which in fact are found from the given Logarithms by subtraction.

Differences.					Logarithms.	Natural Nos.
Fifth.	Fourth.	Third.	Second.	First.		
					33232,52100,17169	2105
				103035512600		
			*243871263		33242,82455,29769	2110
		1151695		102791641337		
	8138		242719568		33253,10371,71106	2115
75		1143557		102548921769		
	8063		241576011		33263,35860,92875	2120
75		1135494		102307345758		
	7988		240440517		33273,58934,38633	2125
		1127506		102066905241		
			239313011		33283,79603,43874	2130
				101827592230		
					33293,97879,36104	2135

We have next to find the mean differences. Multiplying the given first differences by 2, and cutting off the last figure, we get the first mean differences. The remaining mean differences will be found by multiplying the other differences by 4, 8, 16, 32, &c., and cutting off 2, 3, 4, 5, ... figures from the products.

Then these mean differences are to be corrected in the following manner:

The two highest differences—the fifth and the fourth—cannot be corrected, because the seventh and sixth are nothing: for every correction of the differences is made by subtracting the alternate corrected differences of the higher orders: thus, the subtraction of the seventh differences corrects the fifth: that of the sixth, corrects the fourth, &c. Therefore in the present case, the fourth and fifth *mean* differences are taken for the fourth and fifth *corrected* differences.

Every third mean difference however is corrected by subtracting from it three times the fifth corrected difference.

* The numbers here printed in antique type are not inserted in the original; but they have been added to make the author's process more easily followed, in conformity with his remark made further on, p. 81.

Mean Differences.			
First.	20558328267	4	× ·2
	20509784353	8	
	20461469151	6	
	20413381048	2	
Second	9708782	72	× ·04
	9663040	44	
	9617620	68	
Third	9213	560	× ·008
	9148	456	
	9083	952	
	9020	048	
E Fourth	13	0208	× ·0016
	12	9008	
	12	7808	
Fifth		02400	× ·00032
		02400	

9213	560	third mean difference
	72	three times the fifth corrected difference
9213	488	third corrected difference
9148	456	third mean „
	72	three times the fifth corrected „
9148	384	third corrected difference
		D
9083	952	third mean „
	72	
9083	880	third corrected „
9020	048	
	72	
9019	976	third corrected „

From the second mean difference we must subtract twice the fourth corrected difference—and we must moreover take $\frac{7}{5}$ $(1\frac{2}{5})$ of the sixth difference, if any sixth differences have been found in the work.

Third corrected	9213	5	D
	9148	4	
	9083	9	
	9020	0	
Second corrected	9708756	7	C
	9663014	6	
	9617595	1	
First corrected	20558319053	9	B
	20509775205	4	
	20461460067	7	
	20413372020	2	

9708782	72	second mean difference	
26	04	twice the fourth corrected difference	
9708756	68	second corrected difference	
9663040	44	second mean „	
25	80	twice the fourth corrected „	C
9663014	64	second corrected difference	
9617620	68		
25	56	twice the fourth corrected „	
9617595	12	second corrected difference	

From each first mean difference we must deduct the corresponding third corrected difference and $\frac{1}{5}$ of the fifth difference.

20558328267	4	first mean difference
9213	5	third corrected difference
	0048	$\frac{1}{5}$ of the fifth „
20558319053	9	first corrected „
20509784353	8	first mean „
9148	4	third corrected „
20509775205	4	first corrected „
20461169151	6	
9083	9	third corrected „
20461460067	7	first corrected „
20413381048	2	
9020	0	
20413372028	2	first corrected „

$\frac{1}{5}$ of the fifth falls outside the limits and may therefore be safely neglected.

B

In this manner then have all the differences been corrected and prepared for use. If there were more orders of differences, we should proceed in the same way, commencing with the highest orders, which we always suppose to be the least.

The following Table shows what multiple of each difference is to be subtracted in each case:

TABLE X.

A	B	C	D	E	F	G	H	I
20								
19								
18	18(20)							
17	17(19)							
16	16(18)	123·2(20)						
15	15(17)	108·0(19)						
14	14(16)	93·8(18)	400·4(20)					
13	13(15)	80·6(17)	317·2(19)					
12	12(14)	68·4(16)	246·4(18)	629·64(20)				
11	11(13)	57·2(15)	187·0(17)	431·20(19)				
10	10(12)	47·0(14)	138·0(16)	283·80(18)	434·40(20)			
9	9(11)	37·8(13)	98·4(15)	177·84(17)	236·88(19)			
8	8(10)	29·6(12)	67·2(14)	104·72(16)	118·72(18)	111·248(20)		
7	7 (9)	22·4(11)	43·4(13)	56·84(15)	53·20(17)	36·680(19)		
6	6 (8)	16·2(10)	26·0(12)	27·60(14)	20·40(16)	10.760(18)	4·080(20)	
5	5 (7)	11·0 (9)	14·0(11)	11·40(13)	6·20(15)	2·280(17)	·500(19)	
4	4 (6)	6·8 (8)	6·4(10)	3·64(12)	1·28(14)	·272(16)	·032(18)	·0016(20)
3	3 (5)	3·6 (7)	2·2 (9)	·72(11)	·12(13)	·008(15)		
2	2 (4)	1·4 (6)	·4 (8)	·04(10)				
1	1 (3)	·2 (5)						

The numbers placed in column A denote the mean differences of the first, second, third and other orders up to the 20th. But the numbers in the columns B, C, D, &c., show what multiples of each corrected difference are to be subtracted* from those mean differences which are placed in column A in the same line with them. For example: from the sixth mean difference we must subtract six times the eighth corrected difference; $16\frac{2}{10}$ of the 10th corrected difference; 26 times the 12th corrected difference; and so on; and in the same manner from the first mean difference we must subtract the third corrected difference and $\frac{1}{5}$ of the fifth difference.

* Not only for Logarithms are all to be subtracted, but also for Tangents, Secants, and for any the same powers of equidistant numbers. For Sines, however, the differences contained in columns B, D, F, H, are to be added to the mean differences placed in column A: but the others in columns C, E, G, I, are to be subtracted.

Having found these corrected differences, the next step will be to insert each conveniently in its place, in order that in so complicated an operation all confusion may as far as possible be avoided. We shall accomplish this more readily if we have a sheet of cross-ruled paper divided as in the following Table, and if the first, third, fifth, seventh, and other odd differences are written in a different coloured ink from the others.* The given Logarithms marked A occupy every fifth place. The second corrected differences, C, the fourth, E, the sixth, the eighth, &c., are placed to the left in the same line as the Logarithms. But the first corrected differences, B, the third, D, the fifth, seventh, &c., are placed in the centre of each space. Lastly, the vacant places are to be filled up, beginning from the left. By the addition of the fourth differences, we obtain the third: by the addition of the third, we obtain the second; and so on: and in the process of addition we may either add or subtract a unit in the last place, as required. For with irrational quantities, it will be sufficient to have differences approximately true, since we cannot find the true values exactly. For this reason, although I said in the beginning of this chapter that the last figure was to be cut off from the products of the first differences by 2, yet here I have cut off none; but, in the first and remaining differences, I have thought it better to retain one figure beyond the established limits, in order that the work may proceed with greater certainty of accuracy. I recommend the same course to be pursued with Tangents, Secants, and Sines; but in dealing with the powers of equidistant numbers, where the given numbers and all the differences are rational, all may be contained within the prescribed limits; for there always exists a definite number of orders of differences, which cannot be exceeded, when the difference between the numbers is constant. For instance, in squares there are two orders of differences; in cubes, three orders; in the fourth powers, four orders; and so on. And the differences of the highest order are always equal to each other, and equal to the product of the same power of the common difference into the continued product of the index of that power into all lower numbers [equal to $n(n-1)(n-2)\ldots 2.1.b^n$ if the numbers are a^n, $(a+b)^n$, $(a+2b)^n\ldots$] so that if the difference of the given numbers is 1, the last differences will be in the case of squares, 2; in cubes, 6; in the fourth powers, 24; in (5), 120; in (6), 720; in (7), 5040; &c.; these numbers being the continuous products $1.2=2$, $1.2.3=6$, $1.2.3.4=24$, &c. But if the difference of the given numbers be 3, the difference of

* These differences are here printed in antique type.

the highest order will be in the case of the squares 18,—the product of the square 9 into 2; in the case of the cubes, 162,—the product of the cube 27 into 6; in the case of the fourth powers, 1944,—the product of the fourth power 81 into 24, &c.

4th Differences.		2nd and 3rd Differences.				Logarithms and 1st Differences.					Natural numbers.
			9213	5	D	2	05583	19053	9	B	
		97	27144	5							
			9200	4		2	05485	91909	5		
		97	17944	1							
			9187	4		2	05388	73965	4		
E 13	0	97	08756	7	C	33253	10371	71106		A	2115
			9174	4		2	05291	65208	7		
		96	99582	4		255	15663	36315			16
			9161	4		2	05194	65626	3		
		96	90421	0		257	20858	01941			17
			9148	4	D	2	05097	75205	4	B	
		96	81272	6		259	25955	77146			18
			9135	5		2	05000	93932	7		
		96	72137	1		261	30956	71079			19
			9122	6		2	04904	21795	6		
E 12	9	96	63014	6	C	33263	35860	92875		A	2120
			9109	7		2	04807	58781	0		
		96	53905	0		265	40668	51656			21
			9096	8		2	04711	04876	0		
		96	44808	2		267	45379	56532			22
			9083	9	D	2	04614	60067	7	B	
		96	35724	4		269	49994	16600			23
			9071	1		2	04518	24343	2		
		96	26653	3		271	54512	40943			24
			9058	3		2	04421	97689	9		
E 12	8	96	17595	1	C	33273	58934	38633		A	2125
			9045	5		2	04325	80094	8		
		96	08549	6							
			9032	7		2	04229	71545	2		
		95	99516	9							
			9020	0	D	2	04133	72028	2	B	

In all these cases, both in the powers of numbers, and in Logarithms, Tangents and Secants, it will be necessary to include in the

work several more numbers than those between which we interpolate; or we shall not be able to obtain the last differences. Thus, in the example given above, we must take in one direction the numbers 2110 and 2105; and in the other direction 2130 and 2135. But in the case of Sines, if the sines of three equidifferent arcs are given, all the differences, even of the highest order, can be found by the rule of proportion, if required. For the Sines and their Second, Fourth, Sixth, and Eighth differences are always proportional; and the First, Third, Fifth, and Seventh differences are also always proportional. Thus, as the Second differences are themselves proportional to the corresponding Sines; as are also the Fourth, Sixth, &c. differences; so the First, Third, Fifth and Seventh differences are proportional to the cosines of the arcs which are the arithmetic means of the given arcs.

But I feel I have been carried away by these considerations into a longer digression than is warranted by the laws of homogeneous quantities. If you wish to compute another Thousand Logarithms to be added to those I have calculated (suppose the twenty-first Thousand) you must take the fifth part of that number from which you are to begin. The first number will then be 20,000, the fifth part of which is 4000. To the Logarithms of this number and of the next two hundred numbers, add the Logarithm of 5; then the sums will be the Logarithms of each fifth number through the whole Thousand: namely, of 20000, 20005, 20010, 20015, &c. Now their first differences are the same as those of the above two hundred Logarithms; and are found* in the fifth Thousand of my tables. From these differences are to be found the second differences. The second differences will likewise give the third. The fourth differences are however very small, so that we may safely neglect them. Then multiply the first, second and third differences into two [·2], four [·04], eight [·008]. The products will be the mean differences that are to be inserted in their respective places, having first cut off one figure from the second, and two from the third differences. But the first differences are to be kept out of their places until they have been corrected by subtracting the third differences: all the rest are to be obtained by *addition.*

This method of interpolating four Logarithms between two given ones, may be called *Quintisection,* because from one interval five are to be made. General rules can also be given for *Trisection,*

* Briggs's Tables of Logarithms contain, not only the logarithms to 14 decimal places, but also the differences between successive logarithms.

and *Septisection*; but of all these, Quintisection is the best, whether we regard the length or the facility of the computation. Nevertheless it will be worth while to give in a few words the method of Trisection. Take as before the first, second, third, &c., differences of the given quantities. Then divide the first differences by 3, the second by 9, the third by 27, the fourth by 81, and so on; the divisors increasing in triple ratio: and the quotients will be the first, second, third, fourth, &c., *mean* differences. These mean differences are, as before, to be diminished in all cases except in the case of Sines; and then the corrected differences are to be put into their proper places: and, commencing with the differences of the highest order, which are supposed to be the smallest, all the work is to be done as before by addition.

The annexed Table shows how much is to be subtracted from each difference:

1 (12) 1 (11)				
1 (10) 1 (9)	$3\frac{1}{3}$ (12) 3 (11)			
1 (8) 1 (7)	$2\frac{2}{3}$ (10) $2\frac{1}{3}$ (9)	$3\frac{1}{9}$ (12) $2\frac{3}{9}$ (11)		
1 (6) 1 (5)	2 (8) $1\frac{2}{3}$ (7)	$1\frac{6}{9}$ (10) $1\frac{1}{9}$ (9)	$\frac{20}{27}$ (12) $\frac{10}{27}$ (11)	
1 1	$1\frac{1}{3}$ (6) 1 (5)	$\frac{6}{9}$ (8) $\frac{3}{9}$ (7)	$\frac{4}{27}$ (10) $\frac{1}{27}$ (9)	$\frac{1}{81}$ (12)
1 1	$\frac{2}{3}$ (4) $\frac{1}{3}$ (3)	$\frac{1}{9}$ (6)		
A	B	C	D	E

From the first mean difference we must take $\frac{1}{3}$ of the third corrected difference.

From the fourth mean difference we must take $\frac{4}{3}$ of the sixth, $\frac{2}{3}$ of the eighth, $\frac{4}{27}$ of the tenth, $\frac{1}{81}$ of the twelfth corrected differences.

The other Sections, named after the even numbers, as Bisection, Quadrisection, &c., are more difficult. This we also experience in finding the *chords* of circular arcs: for whilst the sections named after the odd numbers show the required chords themselves at one operation; the others, named after the even numbers, develope, not the chords, but only their squares.

Here is an example of Trisection in the Fourth powers.

Mean Differences found by division of the given differences.				Given Differences.				Fourth Powers.	Numbers.
4th by 81.	3d by 27.	2nd by 9.	1st by 3.	4th.	3d.	2nd.	1st.		
								256	4
		$33\frac{45}{81}$		24		302		625	5
	d. $4\frac{72}{81}$		$223\frac{2}{3}$		132		671		
$\frac{24}{81}$		$48\frac{18}{81}$		24		434		1296	6
	d. $5\frac{63}{81}$		$368\frac{1}{3}$		156		1105		
		$65\frac{45}{81}$		24		590		2401	7
								4096	8

The third and fourth mean differences cannot be corrected.

If $\frac{2}{3}$ of the fourth difference be subtracted from the second mean differences, the remainders will be the second corrected differences C.

If $\frac{1}{3}$ of the third difference be subtracted from the first mean differences, the remainders will be the first corrected differences B, as appears from the table X, *ante*.

4th Diffces.	3rd Differences.	2nd Differences.		1st Differences.	4th Powers.	Numbers.
$\frac{24}{81}$		C. $33\frac{29}{81}$			625. A	5
	$4\frac{48}{81}$			$184\frac{7}{81}$		
		$37\frac{77}{81}$			$809\frac{7}{81}$	$5\frac{1}{3}$
	$4\frac{72}{81}$		B	$222\frac{3}{81}$		
	D	$42\frac{68}{81}$			$1031\frac{10}{81}$	$5\frac{2}{3}$
	$5\frac{15}{81}$			$264\frac{71}{81}$		
$\frac{24}{81}$		C. $48\frac{2}{81}$			1296. A	6
	$5\frac{39}{81}$			$312\frac{73}{81}$		
		$53\frac{41}{81}$			$1608\frac{73}{81}$	$6\frac{1}{3}$
	$5\frac{63}{81}$			$366\frac{33}{81}$		
	D	$59\frac{23}{81}$	B		$1975\frac{25}{81}$	$6\frac{2}{3}$
	$6\frac{6}{81}$			$425\frac{56}{81}$		
$\frac{24}{81}$		C. $65\frac{29}{81}$			2401. A	7

We next give a translation of the paper by Legendre in the additions to the *Connaissance des Tems* for 1817, (mentioned by M. Maurice), in which he demonstrates the reasons of the rules laid down by Briggs.

We are indebted to Henry Briggs, Professor of Geometry at Oxford, for two fundamental works, the *Arithmetica Logarithmica* published at London in 1624, and the *Trigonometria Britannica*

published at Gouda in 1633. Each of these works is prefaced by a treatise in which the author has explained, with all necessary details, the various methods employed by him in constructing his Tables. These methods are principally his own invention, and prove him to have been quite familiar with the theory of differences, although he was not acquainted with the general formula for interpolating intermediate values of a function in a series of values corresponding to equidistant values of the argument.

Briggs supplied the place of this formula by a very remarkable method, which may be called the *Method of Quintisection,* by means of which, if a series of equidistant values are given, we may interpolate between any two adjacent values, four others, so that the total number of the terms of the series shall be five times as many as before. In this manner, Briggs extended by successive steps the various tables he wished to construct, until the scheme he had proposed to himself was completed. He does not however give any demonstration of this method, but simply explains the process in the clearest manner, giving numerous illustrations in both his above-mentioned works.

It does not appear that this method has ever attracted much attention, or that any one has tried to demonstrate it. If however we consider that these very works of Briggs's have been the foundation of all or nearly all the Trigonometrical Tables hitherto published, that it is only by means of the Tables they contain that we can, without great labour, find the logarithm of a number, or of a sine, to 14 places of decimals, and a natural sine to 15 places; it will probably not be thought surprising that I have examined with some interest one of the principal bases on which those two great works have been constructed.

I will now proceed with the demonstration at which I have arrived. It is not so simple as I could have wished; but it may perhaps lead to the discovery by some other mathematician of a demonstration more akin to that which the author himself must have discovered, although he neglected to publish it.

Given a series of values of a function, $a, a', a'', a''', \ldots;$ corresponding to the values of the argument $0, 1, 2, 3, \ldots;$ and such, that their first, second, third, &c., differences constantly diminish and at last become so small that they may be neglected; it is required to interpolate four equidistant values between any adjacent two of the given values, so that the terms of the resulting series shall correspond to arguments differing by $\frac{1}{5}$.

By means of this interpolation which quintuples the number of terms, we shall have the new series:

$$y, y', y'', y''', y^{\text{IV}}, y^{\text{V}}, y^{\text{VI}}, y^{\text{VII}}, y^{\text{VIII}}, y^{\text{IX}}, y^{\text{X}}, \ldots;$$

corresponding to arguments

$$0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, 2, \ldots;$$

and the terms $y, y^{\text{V}}, y^{\text{X}}, \ldots$, corresponding to the integral arguments 0, 1, 2, ..., will be the same as the given values $a, a', a'', \ldots$.

I now remark, that according to the known formula for interpolation, we have

$$y'-y=\delta y=a[(1+\delta)^{\frac{1}{5}}-1]$$

provided that after expanding $(1+\delta)^{\frac{1}{5}}$, we change the products $a\delta, a\delta^2, a\delta^3, \ldots$, into successive differences, $\delta a, \delta^2 a, \delta^3 a, \ldots$. Putting $(1+\delta)^{\frac{1}{5}}=\omega$, we shall have

$$\delta y=a(\omega-1).$$

Under the same supposition we shall have, $y'=a\omega$, $y''=a\omega^2$, $y'''=a\omega^3, \ldots$. We may thus form the following table, which contains the symbolical values of the terms $y', y'', y''', \ldots$, and of their successive differences.

Terms.	1st Differences.	2nd Differences.	3d Differences.	4th Differences.	&c.
$y=a$	$\delta y=a\ (\omega-1)$	$\delta^2 y=a\ (\omega-1)^2$	$\delta^3 y=a\ (\omega-1)^3$	$\delta^4 y=a\ (\omega-1)^4$	&c.
$y'=a\omega$	$\delta y'=a\omega\ (\omega-1)$	$\delta^2 y'=a\omega\ (\omega-1)^2$	$\delta^3 y'=a\omega\ (\omega-1)^3$	$\delta^4 y'=a\omega\ (\omega-1)^4$	&c.
$y''=a\omega^2$	$\delta y''=a\omega^2(\omega-1)$	$\delta^2 y''=a\omega^2(\omega-1)^2$	$\delta^3 y''=a\omega^2(\omega-1)^3$	$\delta^4 y''=a\omega^2(\omega-1)^4$	&c.
$y'''=a\omega^3$	$\delta y'''=a\omega^3(\omega-1)$	$\delta^2 y'''=a\omega^3(\omega-1)^2$	$\delta^3 y'''=a\omega^3(\omega-1)^3$	$\delta^4 y'''=a\omega^2(\omega-1)^4$	&c.
$y^{\text{IV}}=a\omega^4$	$\delta y^{\text{IV}}=a\omega^4(\omega-1)$	$\delta^2 y^{\text{IV}}=a\omega^4(\omega-1)^2$	$\delta^3 y^{\text{IV}}=a\omega^4(\omega-1)^3$	$\delta^4 y^{\text{IV}}=a\omega^4(\omega-1)^4$	&c.
$y^{\text{V}}=a\omega^5$	$\delta y^{\text{V}}=a\omega^5(\omega-1)$	$\delta^2 y^{\text{V}}=a\omega^5(\omega-1)^2$	$\delta^3 y^{\text{V}}=a\omega^5(\omega-1)^3$	$\delta^4 y^{\text{V}}=a\omega^5(\omega-1)^4$	&c.
&c.	&c.	&c.	&c.	&c.	&c.

The law of these expressions is evident, and we shall have, generally, whatever may be the values of m and n,

$$\delta^m y^n=a\omega^n(\omega-1)^m;$$

where it is supposed that after having substituted the value $\omega=(1+\delta)^{\frac{1}{5}}$, and expanded the second member by powers of δ, each term as $a\delta^n$, is to be replaced by the difference $\delta^n a$.

All is known when the expansions are completed, but the process is long and troublesome. There is a simpler way of forming

the different values y', y'', , . . by seeking for the relations that may exist between a finite number of them.

With this object, I note that the quantities which may be found directly from the given values are

$$\delta a = a(\omega^5-1),\quad \delta^2 a = a(\omega^5-1)^2,\quad \delta^3 a = a(\omega^5-1)^3,\ \&c.$$

I write the first under the form

$$\delta a = a(\omega-1)(\omega^4+\omega^3+\omega^2+\omega+1)$$

and making $(\omega-1)^2=\omega Z$, which gives

$$\omega^2+1=\omega(Z+2),\quad \omega^4+1=\omega^2(Z^2+4Z+2),$$

I deduce $$\delta a = a\omega^2(\omega-1)(Z^2+5Z+5),$$

or $$\frac{\delta a}{5} = a\omega^2(\omega-1)\left(1+Z+\frac{1}{5}Z^2\right).$$

But if in the table of Differences we introduce the quantity Z, so as not to have any power of $\omega-1$ above the first, we shall have

$\delta y\ \ = a\ \ (\omega-1)$	$\delta^2 y\ \ = a\omega\, Z$	$\delta^3 y\ \ = a\omega\, Z(\omega-1)$	$\delta^4 y\ \ = a\omega^2 Z^2$
$\delta y'\ \ = a\omega\ (\omega-1)$	$\delta^2 y'\ \ = a\omega^2 Z$	$\delta^3 y'\ \ = a\omega^2 Z(\omega-1)$	$\delta^4 y'\ \ = a\omega^3 Z^2$
$\delta y'' = a\omega^2(\omega-1)$	$\delta^2 y'' = a\omega^3 Z$	$\delta^3 y'' = a\omega^3 Z(\omega-1)$	$\delta^4 y'' = a\omega^4 Z^2$
$\delta y''' = a\varpi^3(\omega-1)$	$\delta^2 y''' = a\omega^4 Z$	$\delta^3 y''' = a\omega^4 Z(\omega-1)$	$\delta^4 y''' = a\omega^5 Z^2$
$\delta y^{\text{IV}} = a\omega^4(\omega-1)$	$\delta^2 y^{\text{IV}} = a\omega^5 Z$	$\delta^3 y^{\text{IV}} = a\omega^5 Z(\omega-1)$	$\delta^4 y^{\text{IV}} = a\omega^6 Z^2$
&c.	&c.	&c.	&c.

From this we see that we may write the preceding equation thus:

$$\frac{\delta a}{5} = \delta y'' + \delta^3 y' + \frac{1}{5}\delta^5 y.$$

In the same way the equation $\delta^2 a = a(\omega^5-1)^2$ will become

$$\frac{\delta^2 a}{25} = a\omega^4(\omega-1)^2\left(1+Z+\frac{1}{5}Z^2\right)^2 = a\omega^5 Z\left(1+Z+\frac{1}{5}Z^2\right)^2$$

and expanding the second member, we get

$$\frac{\delta^2 a}{25} = \delta^2 y^{\text{IV}} + 2\delta^4 y''' + \frac{7}{5}\delta^6 y'' + \frac{2}{5}\delta^8 y' + \frac{1}{25}\delta^{10} y.$$

Similarly, by expanding the cube of the trinomial $1+Z+\frac{1}{5}Z^2$, we shall have the equation

$$\frac{\delta^3 a}{125} = \delta^3 y^{\text{VI}} + 3\delta^5 y\ + 3{\cdot}6\delta^7 y^{\text{IV}} + 2{\cdot}2\delta^9 y''' + {\cdot}72\delta^{11} y'' + {\cdot}12\delta^{13} y' + {\cdot}008\delta^{15} y.$$

These three equations, and those which we should form in the same way for the values of $\frac{\delta^4 a}{5^4}$, $\frac{\delta^5 a}{5^5}$, &c., express the same thing as the Table given by Briggs in the *Arith. Logarith. Ed. Lond.* p. 29 (see above, p. 79); and in the *Trigon. Brit.* p. 38.

The quantities $\frac{\delta a}{5}$, $\frac{\delta^2 a}{5^2}$, $\frac{\delta^3 a}{5^3}$, . . . , are what Briggs calls *mean differences;* they give a first approximation to the differences $\delta y''$, $\delta^2 y^{\text{iv}}$, $\delta^3 y^{\text{vi}}$, . . . but these values require to be corrected by means of the following terms. But, from the nature of the case, the successive differences δy, $\delta^2 y$, $\delta^3 y$, . . . must diminish very rapidly, and it will therefore not be necessary to go beyond that order of differences which may be safely neglected in the series a, a', a'', . . and much more in the series y, y', y'', . . . We may therefore suppose the two last differences of the series δy, $\delta^2 y$, $\delta^3 y$, . . . equal to the two mean differences deduced from the two last terms of the series δa, $\delta a'$, $\delta a''$, . . . We must then correct the other differences, beginning with those of the highest order and ending with the differences $\delta^3 y^{\text{vi}}$, $\delta^2 y^{\text{iv}}$, $\delta y''$. We shall thus have the corrected values of these last differences, and by means of these and the preceding ones we can complete by addition the columns of the differences, and lastly we can form the column of values y, y', y'', y''', . . . This is in fact the method of Briggs, which it was our object to demonstrate, and which, although somewhat complicated in appearance, is rendered perfectly clear by the examples the author has given.

Our readers will now be in a position to judge for themselves of the justice of M. Maurice's strictures on Briggs's method. They will see that there is no possibility of confusion in consequence of the differences of various orders having relation to different terms in the series of given values, provided that the differences are written in the usual way—each difference on a line half way between the two values from which it is obtained. They will also see that M. Maurice in asserting that the values in the table on p. 81 are obtained by *subtraction*, and not by *addition*, as stated by Briggs himself, (see p. 12 of this volume,) overlooked the circumstance that a computer would naturally form the successive values by addition, *commencing from the bottom*; instead of beginning at the top and using subtraction. There can be no doubt, we believe, that Briggs's description of his method will be found perfectly clear by any computer wishing to apply it in practice.—ED. *J. I. A.*

Fourier's Statistical Tables. *By* A. De Morgan, Esq.

It is stated in the notes to Cousin's *éloge,* and has been repeated, that Fourier was placed at the head of a *bureau de statistique* by the prefect of the Seine, the Comte de Chabrol. But there is hardly any authentication. Anything done in the subject by Joseph Fourier, the most powerful working member of the Egyptian commission, the originator of mathematical results of the first order of genius, novelty, and utility, and an experienced public man, must be worthy of attention. But so little is his statistical career known, that the *Biographie Universelle* (Michaud, 1856) only ventures its mention of the connexion of Fourier with the volumes presently described as *suivant plusieurs personnes bien instruits.*

The facts are as follows. After the restoration of 1815, Fourier was of course not in good odour. The king refused him admission to the Academy of Sciences; and he was left to the income of a French *savant* who is disowned by the ruling powers, which in England we express as a 'midshipman's half-pay.' The Comte de Chabrol, who had been prefect of the next department to that which Fourier held, privately instituted a board of statistics for the prefecture of the Seine, to which department he had been removed, and placed Fourier at the head. But his name was not published; it does not appear in any part of the four volumes issued under his superintendence. It may almost be suspected that the prefect had obtained permission to employ Fourier in this way, on condition of strict privacy: and the more because a few years after 1815 the refusal to allow him to be a member of the Institute was withdrawn. Fourier hardly deserved any special disapprobation from the Bourbons. He tried to evade seeing Napoleon on the voyage from Elba, but did not succeed. He accepted a prefecture under his old friend and sovereign, which he resigned on being required to make extensive arrests among the legitimists. But he was actually prefect of Grenoble when Napoleon passed through the place; and this seems to have been the circumstance on which his treatment turned.

The volumes issued by Fourier are almost unknown, and are exceedingly scarce. An accurate description may tend to preserve them from oblivion, by preventing their anonymous character from causing them to become waste waper. My copies were given to me by my friend Mr. Libri, to whom they were given by Fourier himself, and acknowledged as his own.

There is one octavo volume, and three quarto volumes, as follows.

Octavo, 1821. Recherches statisques sur la ville de Paris et le département de la Seine, recueil de tableaux dressés et réunis d'après les ordres de M. le Comte de Chabrol, conseiller d'état, préfet du département, Paris.... C. Ballard, imprimeur du roi.
Quarto, 1823. Recherches.... département [as above]. A Paris. De L'Imprimerie Royale.
Quarto, 1826. (As above.)
Quarto, 1829. (As above.)

No account of the varied contents of these volumes could be given in any permissible space: population, mortality, meteorology, health, supply and consumption, &c. &c., are the subject-matter of the tables. There are papers on the formation of population and life tables, and on the value of the mean of observations. The mathematical treatment amounts only to statement of results; but all is done with the compact elegance of Fourier's mathematical style. I give one table, representing the population of Paris in 1817, with distinction of the sexes and their condition as to marriage. This table, as representing the results of the long war and the sweeping conscriptions, will be of special value even now.

[*Paris*, 1817.]

Ages.	Males.				Females.				Total both Sexes.
	Married.	Single.	Widowers	Total.	Married.	Single.	Widows.	Total.	
0- 5	..	22,656	..	22,656	..	22,909	..	22,909	45,565
5-10	..	20,806	..	20,806	..	22,544	..	22,544	43,350
10-15	..	22,995	..	22,995	65	24,308	..	24,373	47,368
15-20	380	32.229	12	32,621	2,796	32,884	44	35,724	68,345
20-25	4,784	21,740	58	26,582	12,099	24,274	362	36,735	63,317
25-30	12,509	14,287	223	27,019	18,949	15,783	1,225	35,957	62,976
30-40	33,456	13,506	1,041	48,003	36,988	15,419	5,743	58,150	106,153
40-50	30,094	6,298	2,180	38,572	30,242	7,426	9,754	47,422	85,994
50-60	26,791	4,375	3,686	34,852	18,321	4,781	11,476	34,578	69,430
60-70	16,254	2,788	4,001	23,043	8,413	3,228	11,446	23,087	46,130
70-80	3,890	986	2,141	7,017	1,619	1,343	5,768	8,730	15,747
80-90	420	169	449	1,038	102	296	1,226	1,624	2,662
90-100	10	8	24	42	2	14	75	91	133
100-	1	...	...	1	..	1	..	1	2

On the Rate of Interest in Loans repayable by Instalments. By Peter Gray, F.R.A.S., *Honorary Member of the Institute of Actuaries.*

There must at the present time be a vast amount of British capital embarked in loans. Foreign states are constantly holding out their lures with more or less of success to our monied men, and financial and other associations are always ready to take charge of the funds of such of them as prefer investing at home. In this state of matters it is somewhat remarkable that there is nothing to be found in our books on interest on the subject of loans. These are usually—and perhaps intentionally—so complicated with conditions in regard to premiums, discounts, times and modes of repayment, &c., as to render it almost always a matter of extreme nicety to determine the rate paid by the borrower for the accommodation, and that realized by the lenders on their investments.* And yet, as I have just said, in no English work that I am aware of, is there anything to be found having special reference to the subject.

It is of course desirable, in the interests of both borrowers and lenders, that the rates involved in the transactions into which they enter should be pretty accurately known; and doubtless those parties form conclusions for themselves upon the points in question. There is however reason to fear (and ground for this opinion will hereafter be shown) that the conclusions thus arrived at are not always trustworthy, nor such as will be borne out by the results. Now I find that in most, if not all, cases, it is quite practicable to determine the exact rates, at the cost of no great expenditure of time or trouble; and I have thought it might be useful to devote a paper to the subject.

The method I propose is simply to treat the repayments as an annuity whose present value is the sum advanced; and the problem is thus reduced to finding the rate of interest involved in the annuity. It is true that the annuities with which we shall thus have to deal will be, mostly, very different from uniform annuities. Still, in point of fact, it will usually, if not always, be found, that the payments are regulated by laws which may be discovered, and the discovery of which renders them amenable to the resources of analysis. In the present paper I shall illustrate the method of treatment I propose by applying it to two examples.

* These two rates will be the same in regard to a specified loan if the whole of the loan is held by a single individual; but by no means necessarily so if it is held by more than one. This will be seen hereafter.

I have said that I am not aware that the subject of loans is treated in any English work. A French work however, of which I give the title below,* has recently come into my hands, in which it is taken up. It consists of 128 pages of introductory matter and 129 pages of tables. The principal tables are the amount of one franc (or one pound), the amount of an annuity of one franc, and the annuity that one franc will purchase, all for every number of years from one to one hundred, and at no fewer than eighty-five rates of interest, extending from one to ten per cent. No tables approaching these in the number of rates exhibited have hitherto been published. I mention this work here because my first example is the most complex of those solved by M. Violeine, and I shall have occasion to remark on the method he employs, which I consider to be founded on an altogether erroneous principle.

Example 1.†—A loan of £10,000,000 is contracted at 3 per cent.; and the debt is represented by 10,000 bonds, nominally of £1,000 each. These are to be paid off with a premium of 25 per cent., as follows:—94 the first year, 102 the second, 110 the third, and so on, increasing by 8 each year till the fortieth, when the number paid off will be 406, completing the number of 10,000. It is required to determine the rate per cent. that the loan costs the borrower, and also the rate that will be realized by the holders of the bonds.

Here it will be remarked that although the interest to be paid is 3 per cent., the cost of the loan to the borrower is much more. The charge for each year consists, in addition to the interest on the amount due at the beginning, of a premium of 25 per cent. on the portion of principal paid off at the end of it.

The conditions of the loan might have been stated differently. The loan of £10,000,000 is here issued at par, and the sum lent is to be returned with a premium of 25 per cent. But the loan might have been called one of £12,500,000 issued at 20 per cent. discount, bearing interest at the rate of 2·4 per cent., (125 : 120 :: 3 : 2·4) and repayable at par. But borrowers are wise in their generation. There is little doubt that although the two schemes are identical, to many lenders the first would prove more attractive than the second. There is a charm in the prospect

* *Nouvelles Tables pour les Calculs d'Intérêts Simples et Composés, d'Amortissement, d'Annuités de Primes, etc. Par P.-A. Violeine. A Vaugirard*, 1854. 4to. pp. 128 and 130. I may add that this work appears to be a recognised authority for the purposes of the *Credit Foncier*. A copy of it, I learn, has been recently added to the Library of the Institute of Actuaries.

† Violeine, p. 124. M. V. has *francs*. I use *pounds*, as a measure of value with which we are more familiar.

of getting back one's capital not only intact, but increased by 25 per cent., (a moderate interest being paid upon it while withheld,) which would not be found in a discount of 20 per cent. at the outset, accompanied as it is by an abatement of 0·6 per cent. in the rate of interest allowed, and repayment at par.

It may be here remarked also, that the object, and the effect, of the annual increase in the number of bonds paid off, is to equalize, in a measure, the annual payments of the borrower. As the debt is reduced, the interest payable decreases; and the power of redemption, out of a uniform or nearly uniform revenue, consequently increases. Although it is no part of the *quæsita* of the problem to show how, with the end which has just been specified in view, the particular gradation 94, 102, &c., has been obtained, it may nevertheless be of use to point this out.

To pay off £10,000,000 in forty years, at say 3 per cent., we find by reference to the tables, an annual payment of £432,624 would be necessary. Consequently, £300,000 being the interest payable the first year, £132,624 remains available for the reduction of the principal; and this would suffice to pay off about 106 bonds, at £1,250 each.* Let a be the number of bonds to be paid off the first year, and d the annual increase, Then the sum of forty terms of the series, a, $a+d$, $a+2d$, ... is, by a known theorem,

$$40a+\frac{40\times 39}{2}d=40a+780d.$$

Equating this to 10,000, the entire number of bonds, we get,

$$a=250-19\tfrac{1}{2}d.$$

We must here substitute for d the least *even* number that will give for a a value not exceeding 106. This is easily found to be 8; whence $a=94$, and the series is 94, 102, 110 .... 406.

To proceed now to the solution, first as regards the cost of the loan to the borrower. I have elsewhere shown† that if b_1, b_2, b_3, &c., denote the successive payments of an annuity, the present value of that annuity for n years, will be,

$$\frac{b_1}{i}+\frac{\Delta b_1}{i^2}+\frac{\Delta^2 b_1}{i^3}+\ldots-v^n\left(\frac{b_{n+1}}{i}+\frac{\Delta b_{n+1}}{i^2}+\frac{\Delta^2 b_{n+1}}{i^3}+\ldots\right),\ddagger\ \ldots\ (1.)$$

* The above is a very rude approximate process.

† *Assurance Magazine*, vol. vi., p. 191. See also Introduction to the re-issue of Orchard's *Assurance Premiums*, pp. 12, 13.

‡ The above expression is simply the finite integral of $b_n v^n$. The general form is,

$$\Sigma b_n v^n = C+\frac{b_1}{i}+\frac{\Delta b_1}{i^2}+\frac{\Delta^2 b_1}{i^3}+\ldots$$

For $C=b_0$, this gives the present value of the annuity to infinity when the first payment is made now; and for $C=0$, it gives the like when the first payment is made a year

where i is the rate of interest per pound, and $v=(1+i)^{-1}$, as usual. And this will obviously be a finite expression if b_m be a rational and integer function of m, since in that case the differences of b_1, and b_{n+1}, will ultimately vanish.

This theorem will serve our present purpose. We shall have first to determine the law of the annuity in order to be able to assign the values of the symbols, b_1, Δb_1, &c.; then, substituting these in (1), to equate the result to 10,000,000, the given present value. The value of i deduced from the equation so formed, will be the rate that the loan costs the borrower.

To determine the law of the annuity we must form the numerical values of the first few of its terms, as follows:—

		10,000,000	
Int. at 3 per cent.	300,000		
Pm. on 94 Bds.	23,500	323,500	Charge, 1st year
		10,323,500	
Int.	300,000		
94 Bds. . . .	117,500	417,500	Payment, 1st year, $=b_1$
		9,906,000	
Int.	297,180		
Pm. on 102 Bds.	25,500	322,680	Charge, 2nd year
		10,228,680	
Int.	297,180		
102 Bds. . .	127,500	424,680	Payment, 2nd year, $=b_2$
		9,804,000	
Int.	294,120		
Pm. on 110 Bds.	27,500	321,620	Charge, 3rd year
		10,125,620	
Int.	294,120		
110 Bds. . .	137,500	431,620	Payment, 3rd year, $=b_3$
		9,694,000	
Int.	290,820		
Pm. on 118 Bds.	29,500	320,320	Charge, 4th year
		10,014,320	
Int.	290,820		
118 Bds. . .	147,500	438,320	Payment, 4th year, $=b_4$
		9,576,000	

hence. The form in the text is that which arises when the integral is taken between the limits 1 and $n+1$. Multiplication of it by $(1+i)^n$ converts it into

$$(1+i)^n\left(\frac{b_1}{i}+\frac{\Delta b_1}{i^2}+\ldots\right)-\left(\frac{b_{n+1}}{i}+\frac{\Delta b_{n+1}}{i^2}+\ldots\right)$$

which is the *amount* of the annuity, first payment a year hence, in n years.

Differencing the above values of b_1, b_2, &c., as below:—

$$\begin{array}{l|l|l|l} b_1=417500 & 7180 & -240 & 0 \\ b_2=424680 & 6940 & -240 & \\ b_3=431620 & 6700 & & \\ b_4=438320 & & & \end{array}$$

we have, $b_1=417500$, $\Delta b_1=7180$, $\Delta^2 b_1=-240$, $\Delta^3 b_1=0$ ...
Now we know that

$$b_{1+m}=b_1+m\Delta b_1+\frac{m(m-1)}{2}\Delta^2 b_1+\ldots$$

therefore, putting in the above values,

$$b_{1+m}=417500+7300m-120m^2;$$

and finally, changing m into $m-1$,

$$b_m=410080+7540m-120m^2.$$

This is the general expression for the mth payment; but we shall find that the expression for the $(1+m)$th will answer our purpose rather more conveniently.

The theorem (1) for $n=40$ becomes,

$$\frac{b_1}{i}+\frac{\Delta b_1}{i^2}+\frac{\Delta^2 b_1}{i^3}-v^{40}\left(\frac{b_{41}}{i}+\frac{\Delta b_{41}}{i^2}+\frac{\Delta^2 b_{41}}{i^3}\right);$$

from which it appears that having formed b_1, Δb_1 and $\Delta^2 b_1$, we have still to form b_{41}, Δb_{41} and $\Delta^2 b_{41}$.

We substitute 40, 41 and 42 successively, for m, in b_{1+m}, as follows:—*

$$\begin{array}{rrl} -120 & 7300 & 417500(40 \\ & 2500 & 517500 = b_{41} \\ & -2300 & \\ \hline -120 & -2300 & 517500(1 \\ & -2420 & 515080 = b_{42} \\ & -2540 & \\ \hline -120 & -2540 & 515080(1 \\ & -2660 & 512420 = b_{43} \end{array}$$

We have in this process b_{41} and Δb_{41}, and we already know that $\Delta^2 b_{41}=-240$. For distinctness however I determine them as follows:—

$$\begin{array}{l|l|l} b_{41}=517500 & -2420 & -240 \\ b_{42}=515080 & -2660 & \\ b_{43}=512420 & & \end{array}$$

We therefore have

$$b_{41}=517500,\ \Delta b_{41}=-2420,\ \text{and}\ \Delta^2 b_{41}=-240.$$

Substituting now in (1) we have for determining the value of i, the equation,

* See *Journal of the Institute*, vol. xiii., p. 64.

$$\frac{417500}{i}+\frac{7180}{i^2}-\frac{240}{i^3}-v^{40}\left(\frac{517500}{i}-\frac{2420}{i^2}-\frac{240}{i^3}\right)=10{,}000{,}000.$$

There being no direct method of solving this equation, the solution must be effected by trial; and in this we shall be aided by the use of logarithms. For this purpose I put the expression into the following more convenient although less elegant form:—

$$\frac{417500}{i}+\frac{7180}{i^2}-\frac{240}{i^3}$$
$$-\frac{517500}{i(1+i)^{40}}+\frac{2420}{i^2(1+i)^{40}}+\frac{240}{i^3(1+i)^{40}}=10{,}000{,}000.$$

The operation is as follows:—

	i	·05	·04	·0368	·03727	·0372585
(1)	log i	$\bar{2}$·6989700	$\bar{2}$·6020600	$\bar{2}$·5658478	$\bar{2}$·5713594	$\bar{2}$·5712254
(2)	„ i^2	$\bar{3}$·3979400	$\bar{3}$·2041200	$\bar{3}$·1316956	$\bar{3}$·1427188	$\bar{3}$·1424508
(3)	„ i^3	$\bar{4}$·0969100	$\bar{5}$·8061800	$\bar{5}$·6975434	$\bar{5}$·7140782	$\bar{5}$·7136762
	„ $(1+i)^{10}$	0·2118930	0·1703334	0·1569499	0·1589182	0·1588699
	„ $(1+i)^{40}$	0·8475720	0·6813336	0·6277996	0·6356728	0·6354796
(4)	„ $[i(1+i)^{40}]$	$\bar{1}$·5465420	$\bar{1}$·2833936	$\bar{1}$·1936474	$\bar{1}$·2070322	$\bar{1}$·2067050
(5)	„ $[i^2(1+i)^{40}]$	$\bar{2}$·2455120	$\bar{3}$·8854536	$\bar{3}$·7594952	$\bar{3}$·7783916	$\bar{3}$·7779304
(6)	„ $[i^3(1+i)^{40}]$	$\bar{4}$·9444820	$\bar{4}$·4875136	$\bar{4}$·3253430	$\bar{4}$:3497510	$\bar{4}$·3491558

„ 417500	5·6206565	5·6206565	5·6206565	5·6206565	5·6206565
(1)	$\bar{2}$·6989700	$\bar{2}$·6020600	$\bar{2}$·5658478	$\bar{2}$·5713594	$\bar{2}$·5712254
A	6·9216865	7·0185965	7·0548087	7·0492971	7·0494311
„ 7180	3·8561244	3·8561244	3·8561244	3·8561244	3·8561244
(2)	$\bar{3}$·3979400	$\bar{3}$·2041200	$\bar{3}$·1316956	$\bar{3}$·1427188	$\bar{3}$·1424508
B	6·4581844	6·6520044	6·7244288	6·7134056	6·7136736
„ 240	2·3802112	2·3802112	2·3802112	2·3802112	2·3802112
(3)	$\bar{4}$·0969100	$\bar{5}$·8061800	$\bar{5}$ 6975434	$\bar{5}$·7140782	$\bar{5}$·7136762
C	6·2833012	6·5740312	6·6826678	6·6661330	6·6665350
„ 517500	5·7139104	5·7139104	5.7139104	5·7139104	5·7139104
(4)	$\bar{1}$·5465420	$\bar{1}$·2833936	$\bar{1}$·1936474	$\bar{1}$·2070322	$\bar{1}$·2067050
D	6·1673684	6·4305168	6·5202630	6·5068782	6·5072054
„ 2420	3·3838154	3·3838154	3·3838154	3·3838154	3·3838154
(5)	$\bar{2}$·2455120	$\bar{3}$·8854536	$\bar{3}$·7594952	$\bar{3}$·7783916	$\bar{3}$·7779304
E	5·1383034	5·4983618	5·6243202	5·6054238	5·6058850
„ 240	2·3802112	2·3802112	2·3802112	2·3802112	2·3802112
(6)	$\bar{4}$·9444820	$\bar{4}$·4875136	$\bar{4}$·3253430	$\bar{4}$·3497510	$\bar{4}$·3491558
F	5·4357292	5·8926976	6·0548682	6·0304602	6·0310554

A	8350000	10437500	11345109	11202041	11205496
B	2872000	4487500	5301867	5168989	5172180
C	1920000	3750000	4815793	4635888	4640182
D	1470200	2694700	3313317	3212759	3215181
E	137500	315000	421038	403110	403539
F	272700	781100	1134666	1072655	1074126
	11632200	16021100	18202680	17846795	17855341
	3390200	6444700	8129110	7848647	7855363
Result	8242000	9576400	10073570	9998148	9999978

The scheme exhibits the details of five trials, each occupying a column, with their results. Each column is headed with the value of i used in that column. The first portion of the column is occupied with the formation of the logarithms of the denominators of the several terms; the second portion shews the formation of the logarithms (marked A, B . . . . F) of the terms; and the third portion exhibits the combination of the numbers corresponding to those logarithms for the production of the result, which occupies the last line. The positive and the negative terms are distinguished from each other by being printed in different type.

I commence my trials, pretty much at random, with 5 per cent. This is found to be too great, the result, 8242 . . . , being too small. I try 4 per cent.; but this too appears to be too great, although nearer the truth than 5 per cent. I now apply the rule of False Position, by assuming that variation in the rate is approximately proportional to variation in the result.

Thus:—

$$\begin{array}{rr} 95764 & 100000 \\ 82420 & 95764 \\ \hline 13344 & : 4236 :: -\cdot01 : -\cdot0032. \end{array}$$

We thus find that since a variation of $-\cdot01$ in the rate produces in the result a variation of 13344, therefore 4236, the required variation in the result, will be produced by a variation in the rate of $-\cdot0032$. Hence $\cdot04-\cdot0032=\cdot0368$ is the next rate to be tried. This is found to be nearer the truth, but too small. A new correction is then obtained as follows:—

$$\begin{array}{rr} 100735 & 100000 \\ 95764 & 100736 \\ \hline 4972 & : -736 :: -\cdot0032 : \cdot00047. \end{array}$$

Hence $\cdot0368+\cdot00047=\cdot03727$ is the corrected rate. This, as appears by the result, is a little too great. We therefore seek another correction, as follows:—

999815	1000000
1007357	999815
—7542 :	185 : : ·00047 : —·0000115.

And ·03727 — ·0000115 = ·0372585 is found on trial to be the rate far more than sufficiently near to the truth to serve any practical purpose.

We conclude then that 3·72585, or, rather more nearly, 3·72584, is the rate per cent. that the loan costs the borrower; or, in other words, that it is the rate at which the sum borrowed must be improved, so as just to provide for the stipulated payments as they fall due.

I may here add that the foregoing result has been fully verified by a different method—in fact it was obtained before the problem was brought under my notice—by my friend Colonel Oakes.

Instead of the value just found, M. Violeine assigns as the exact rate—*le taux exact*—3·7613, which differs from it materially. He effects his solution by means of a general formula, of which he offers no demonstration. Here it is:—

$$\text{“R}=\frac{(pr+af+aft)\dfrac{b^n-1}{b-1}-\dfrac{afr-df(1+t)}{b-1}\left(\dfrac{b^n-1}{b-1}-n\right)-df\left\{\dfrac{1}{b-1}\left[\dfrac{b^n-1}{b-1}-b-(n-1)\right]-\dfrac{n^2-n-2}{2}\right\}}{\dfrac{b^n-1}{b-1}}$$

“ *dans laquelle*

“ p = *la somme empruntée*, ... [10,000,000]

“ r = *le taux de l'intérêt* °/。

“ $b=1+\dfrac{r}{100}$,

“ f = *la valeur d'une action ou obligation*, ... [1,000]

“ a = *nombre d'actions remboursées la première année*, . . [94]

“ d = *la raison de la progression*, ... [8]

“ t = *taux* °/。 *de la prime*, ... [25]

“ R = *paiement annuel.*”

On this it is necessary to remark at the outset, that although r and t are here defined as rates *per cent.*, they are used in the formula as rates *per pound.* Also, n, the number of years, is omitted from the enumeration.

Now it will be found that on inserting the values belonging to the present problem (which I have placed opposite the several symbols),

* Surely the first term of this numerator were better written $\{pr+af(1+t)\}$, and the last $\dfrac{(n-2)(n+1)}{2}$.

and any assumed value of r, the numerator of this singular expression gives the amount at the end of the term, 40 years, at the assumed rate, of the variable annuity, that is of the payments that have to be made by the borrower; and the denominator is the amount in the same time of an annuity of £1. Hence the entire expression, the value of R—the *paiement annuel*, as M. V. calls it —is the uniform annuity, at the same rate and during the same term, that is equivalent to the variable annuity. Now we know that the variable annuity amortizes—redeems, pays off—the loan in 40 years at a certain rate of interest; and consequently, by what has just been shewn, the value of R corresponding to that rate, would be the uniform annuity which would amortize the loan in the same time. Hence if we know this rate we shall be able, if we please, to determine R, the uniform amortizing annuity. But we do not know this rate. It is in fact the very rate we are in search of; and all we know about it is that it is greater than 3 per cent., since, in addition to interest at this rate, it has to include the equivalent of the premium of 25 per cent. on the sum borrowed. And therefore we cannot (in this way) arrive at the value of R. M. Violeine is not arrested by this difficulty. He assumes 3 per cent. for the rate, and calls the resulting value of R, 487,424, the uniform amortizing annuity. He then finds (by a method that I need not here stop to describe) that the annuity 487,424 will amortize 10,000,000, the sum borrowed, in 40 years at 3·7613 per cent;—in other words, that 10,000,000 is the value of an annuity of 487,424 for 40 years at 3·7613 per cent. And therefore he assigns this rate as the rate required: not an approximation to it, but the *exact* rate—*le taux exact.*

It is not easy to unravel the tangle here. It is however plain enough that, 487,424 not being the correct amortizing annuity, the rate found cannot be the true rate; and the valuing of the annuity at two different rates is entirely analogous to giving different values to the same symbol on opposite sides of an equation. One is in fact irresistibly reminded of Hogarth's well known illustration of false perspective, in which we see, amongst other incongruities, a man on a hill in the distance lighting his pipe at a candle held by a woman from the first (or second) floor window of a house in the foreground.

I now proceed to the second requirement of the problem, namely, the determination of the rate of interest that will be realized by the lenders. Were there but one lender—were the whole of the bonds held by a single individual—the rate he would

realize would obviously be, as already remarked, that paid by the borrower. But the bonds are not, either necessarily, or in point of fact, so held. There *may* be as many lenders as there are bonds; and the rate realized on each bond will depend on the period at which it is paid off, since it is then that the premium becomes available. What we have to do therefore is to find the rate that will be realized on a bond according as it falls to be paid off in the first, second, third, &c., year. I have omitted to mention that the bonds to be paid off each year are selected by lot; and as the rate varies between wide limits—being no less than 28 per cent. on the bonds paid off in the first year, while it is but a little over 3 per cent. on those paid off in the last—it is quite conceivable that this element of uncertainty—embracing as it does the *possibility* of a very large return on the sum invested—may act with no small effect as a lure on minds of a certain constitution.

Take a bond which will be paid off in the nth year. In return for his £1000 invested now the holder receives an annuity of £30, for n years, and a sum of £1250 at the end of the nth year. The present value of the annuity, at i per pound, is

$$\frac{30(1-v^n)}{i},$$

and that of the sum is, $1250v^n$.

Hence $$\frac{30(1-v^n)}{i}+1250v^n=1000,$$

or $$\frac{3-3v^n+125iv^n}{i}=100,$$

or $$\frac{3+(125i-3)v^n}{100i}=1.$$

And from this equation we shall have to determine the values of i for every value of n from 1 to 40.

Like the former equation the present can only be solved by trial; but although the labour of doing so in any particular case is not great, I have not thought it worth while to encounter it in more than four instances, viz., for $n=10$, 20, 30, and 40. For $n=1$ we have $100i=28$ independently of the theorem. In the following table the first column contains the rates per cent. as determined by the theorem, and the second contains the rates assigned by M. Violeine for the same values of n.

1	28·	28·
10	4·989	4·77
20	3·8526	3·67
30	3·495	3·34
40	3·307	3·19

For illustration I subjoin the operation for $n=20$.

$100i$	4·00	3·5	3·857	3·8526
$125i$	5·00	4·375	4·82125	4·81575
$125i-3$	2·00	1·375	1·82125	1·81575
log $(1+i)^{10}$	0·1703334	0·1494035	0·1643577	0·1641738
„ $(1+i)^{20}$	0·3406668	0·2988070	0·3287154	0·3283476
„ v^{20}	$\bar{1}$·6593332	$\bar{1}$·7011930	$\bar{1}$·6712846	$\bar{1}$·6716524
„ $(125i-3)$	0·3010300	0·1383027	0·2603696	0·2590561
Sum	$\bar{1}$·9603632	$\bar{1}$·8394957	$\bar{1}$·9316542	$\bar{1}$·9307085
$(125i-3)v^{20}$	·912774	·691028	·854386	·852528
$3+(125i-3)v^{20}$	3·912774	3·691028	3·854386	1 852528
log „	0·5924849	0·5671474	0·5859552	0·5857458
„ $100i$	0·6020600	0·5440680	0·5862496	0·5857539
Diff.	$\bar{1}$·9904249	0·0230794	$\bar{1}$·9997056	$\bar{1}$·9999919
Result.	·97819	1·05458	·99932	·9999815

The principle of this process is the same as that already employed. We first try $i=\cdot04$, which is found to be too great. We then try $i=\cdot035$, which is too small. A correction to this last value is then found thus:—

$$\begin{array}{ll} 1\cdot05458 & 1\cdot00000 \\ \cdot97819 & 1\cdot05458 \\ \hline \cdot07639 : -\cdot05458 & :: \ -\cdot005 : \cdot00357. \end{array}$$

We thus have $\cdot035+\cdot00357=\cdot03857$ for a new value, which is too great, and we seek another correction as follows:—

$$\begin{array}{ll} \cdot99932 & 1\cdot00000 \\ 1\cdot05458 & \cdot99932 \\ \hline -\cdot05526 : \cdot00068 & :: \ \cdot00357 : -\cdot0000439. \end{array}$$

This gives $i=\cdot03857-\cdot000044=\cdot038526$, which is very near the truth, the error being $1-\cdot9999815=\cdot0000185$. I have, for exemplification of the principle, gone much further than is necessary. The value ·03857 is quite near enough, and that was reached by two operations.

It will have been perceived that in the solution of this part of the problem, as in that of the former, I am at variance with M. Violeine. His results, except in the case of $n=1$, are always less than mine. Let us attend then to his method of solution. It is so exceedingly simple that it will be matter for regret to find that it gives erroneous results. M. Violeine here, as in the former case, selects the end of the term, (that is, the date of the payment of a bond) as the epoch of comparison; being probably led to this by the circumstance that his tables have reference to *amounts*, and not *present values*. The bondholder having during the term

received annual interest at 3 per cent., and being, at the end of that period, put in possession of the principal with the premium upon it, M. V. considers that the value of the whole at that date is, first, the amount of the bond improved at 3 per cent. for n years, viz., $1000(1\cdot03)^n$; and secondly, the premium just paid, viz., £250. So that, calling i the required rate, we have

$$1000(1+i)^n=1000(1\cdot03)^n+250,$$

or,

$$(1+i)^n=(1\cdot03)^n+\cdot25;$$

whence,

$$\log(1+i)=\frac{\log\{(1\cdot03)^n+\cdot25\}}{n},$$

And from this equation the value of i for every value of n can be very readily determined.

M. Violeine has determined the whole of the values; and I have selected five of those which he gives for comparison with the corresponding values deduced by my process. With the exception of the first, which is the same, M. V.'s values are always less than mine. And the reason is that in his process it is implied that the payments of the annuity as received are forthwith invested at 3 per cent., while in my process no such unwarranted assumption is necessary. Money moreover is, by the terms of the problem, worth more than 3 per cent. To procure the required accommodation *nominally* at 3 per cent., the borrower, in addition to this rate, has to pay also a premium of 25 per cent. on the amount lent.

On the whole, in the view of M. Violeine's method of going to work, I think I may venture to remark, that if our Gallican neighbours have been the first to enter on this new field, their success in the cultivation of it has not hitherto been such as need unduly excite our national susceptibilities.

The second example I had in view was the Austrian Loan of 1865; but in consequence of the space already occupied the consideration of this must be deferred.

Conditions of Assurance.

FOR many years past there has been a continual tendency on the part of our Life Insurance Companies to make the conditions of their policies as to foreign residence and travel more simple and more liberal. Formerly, persons whose lives were insured were not allowed to proceed out of Europe without special permission; and the sea passage to the continent must be made within certain specified limits—usually the Texel and Brest. Some remarks will

be found on this subject in Mr. Babbage's "Comparative view of the various Institutions for the Assurance of Lives" (1826); and it is curious to note the immense changes which have taken place since that date in the manner of conducting the business of Life Assurance.

Latterly, many Offices have allowed the lives assured to travel or reside without extra charge in any part of the world distant more than a certain number of degrees—31, 33, or 35—from the Equator. Such a regulation has the merit of simplicity; and although under it the life insured may proceed to high latitudes, and to certain other localities where there is a greatly increased risk of death, yet the cases where that will happen, will probably always be very rare.

We notice that another step in the same direction has lately been taken (under the advice of Mr. Bailey) by an Office which has issued a table of premiums "for the insurance of £100 for the "whole term of a single life, *with permission, in time of peace, to* "*reside in any part of the world.*" The extra premium according to these tables is least at the ages 30 to 35; and in the case of "without profit" policies, ranges from about £1. 6*s.* per £100 at those ages, to £1. 13*s.* 9*d.* at 20, in the one direction; and to £1. 19*s.* 8*d.* at the age of 60, in the other. The object of these tables is thus explained.

"No extra charge is made for sea risk, except in the case of seafaring men.

"In time of peace, residence is allowed, without extra charge, in any part of Europe, also in Egypt, the Holy Land, Madeira, the Australian Colonies, the Cape of Good Hope, Natal, and in any other part of the world North of 33° of North Latitude, or South of 31° of South Latitude. For all other parts of the world one uniform scale of Premiums has been constructed, which will be found considerably lower than the rates commonly charged for foreign residence. By this arrangement any one contemplating Life Assurance may know what outlay he will have to incur, should circumstances require him to go Abroad at any future time.

"For example—A, having assured his life at the age of 30 for £1,000, under Table 3, pays an annual premium of £25. 6s. 8d.; five years afterwards, having occasion to go to and reside in Jamaica, the annual premium will be increased to £43. 6s. 8d. Vide Table 3a, age 35."

On these conditions we may remark that we fully concur in the propriety of abandoning the extra premium for sea risk.

But the new scheme as to foreign residence appears open to some objections. It will be noticed that the Office, after a policy has once been granted, abandons all power of requiring a special premium for residence in any peculiarly unhealthy climate. The effect of this must, in the long run, virtually be, that those of the

assured who do not incur any such special risk, pay for those who do.

On the other hand, it may be a great boon to many of the insured to know beforehand the extreme premium that can be demanded, whatever part of the world they may visit. But this advantage is not so great as it may at first sight appear; for it will be noticed, that when the life assured proceeds for the first time beyond the ordinary limits, the premium to be paid depends only on the then age, and is the same whatever length of time the assurance may previously have been in force. In effect, therefore, the former insurance is cancelled, and no consideration allowed for it—except that on the return of the life within the ordinary limits, the premium will be reduced to the original amount. It appears to us that it would be sufficient under the circumstances supposed, to charge an additional premium equal to the difference between the ordinary and the whole world premiums for the then age of the life assured. In this way, the assured would receive the full benefit of the value of his existing policy.

The following are the whole world premiums in question.

Annual Premiums for the Assurance of £100, for the Whole Term of a Single Life, *with permission, in time of peace, to reside in any part of the world.*

Age next birth-day.	Without Profits.	With Profits.	Age next birth-day.	Without Profits.	With Profits.
	£ s. d.	£ s. d.		£ s. d.	£ s. d.
20	3 9 8	3 17 5			
21	3 9 9	3 17 6	41	4 10 5	5 0 5
22	3 9 9	3 17 6	42	4 12 11	5 3 3
23	3 9 10	3 17 7	43	4 15 8	5 6 4
24	3 9 11	3 17 8	44	4 18 5	5 9 4
25	3 10 1	3 17 10	45	5 1 2	5 12 5
26	3 10 3	3 18 1	46	5 4 0	5 15 7
27	3 10 6	3 18 4	47	5 6 9	5 18 7
28	3 10 10	3 18 8	48	5 9 5	6 1 7
29	3 11 4	3 19 3	49	5 12 4	6 4 10
30	3 12 0	4 0 0	50	5 15 5	6 8 3
31	3 12 10	4 0 11	51	5 18 10	6 12 0
32	3 14 0	4 2 2	52	6 2 7	6 16 2
33	3 15 2	4 3 6	53	6 6 7	7 0 8
34	3 16 6	4 5 0	54	6 11 0	7 5 7
35	3 18 0	4 6 8	55	6 15 8	7 10 9
36	3 19 9	4 8 7	56	7 0 9	7 16 5
37	4 1 7	4 10 8	57	7 6 4	8 2 7
38	4 3 7	4 12 10	58	7 12 3	8 9 2
39	4 5 9	4 15 3	59	7 18 8	8 16 4
40	4 8 0	4 17 9	60	8 5 7	9 4 0

As the above mentioned changes in the conditions necessitated an alteration in the form of the policy, the opportunity was taken to revise it in other respects. And the form now adopted is so much shorter and simpler than those generally in use, that we believe we shall be doing many of our readers a service by inserting it in this *Journal.*

LIFE POLICY.

No.______ **Life of Another.**
£______ **Term, Whole Life.**

BY THE __________________.

Principal Place of Business at which Notices of Assignment may be given in pursuance of "The Policies of Assurance Act, 1867."*

No. ________________ LONDON.

Whereas , . (hereinafter called the Assured) having an interest in the life of . (hereinafter called the Nominee) ha agreed with THE ________________, to effect an Assurance on the Life of the Nominee for the whole term thereof for the Sum of and the Assured and Nominee have respectively signed or caused to be signed on their behalf respectively and delivered to THE __________________ Declarations dated the day of 186 as the basis of such Assurance.

And Whereas the Assured ha paid to THE ________________ the Premium of . being the consideration for such Assurance for One Year from the date hereof and ha proposed to pay the same Premium Annually on the . day of in the next and in every succeeding year during the Life of the Nominee.

Now this Policy witnesseth That if the Nominee shall die within One Year from the date hereof or if the Nominee shall live beyond the said One Year and the Assured shall Yearly upon or within thirty days next after the day of in every succeeding year during the Life of the Nominee pay to THE ________________ the Annual Premium of then THE ________________ shall pay at the principal Office in London of THE ________________ unto the Assured the Sum of . on the expiration of Three Calendar months next after proof shall be made to the satisfaction of THE ________________ of the death of the Nominee.

Provided always that if the aforesaid Declarations or either of them shall be found to be untrue in any particular or if the Nominee shall die in any part of the World between the thirty-third parallel of North latitude and the thirty-first parallel of South latitude otherwise than except in passing through the same by sea and otherwise than and except in Egypt the Holy Land Madeira the Australian Colonies the Cape Colony and Natal

* Is not this line superfluous? We incline to the opinion that the provision of the Act is satisfied by simply giving the address of the principal place of business.

without having obtained the consent in writing of THE ———————— to reside within the said parallels of latitude respectively or if the Nominee shall enter into any Military Naval or Maritime Service or shall engage in any capacity whatever in actual warfare or shall visit any country or district the seat of War without the previous consent in writing of THE ———————— then and in any such case this Policy and the Assurance hereby made shall be void and the Premiums paid in respect thereof shall be retained by THE ———————— but nevertheless this Policy shall not be avoided if the Nominee shall have done any such act as aforesaid without the knowledge of the Assured and if immediately on becoming acquainted with such act and afterwards in due course the Assured shall have paid the extra Premium which THE ———————— would have required for their consent to such act.

In Witness whereof THE ———————— have caused their Common Seal to be hereunto affixed this day of One Thousand Eight Hundred and Sixty-

BY ORDER OF THE COURT OF DIRECTORS

Secretary.

It will here be noticed that if the life assured transgress the stipulated limits, with the knowledge of the holder of the policy, and the extra premium be not paid, the policy is not thereby rendered void unless the life assured happen to die while beyond limits. Also, that if the life assured so transgress without the knowledge of the holder, and return after any time, long or short, no extra premium can be required from the holder of the policy, although the Office will have been on the risk throughout.

The Sales of Reversions Act, 1867.

BY this Act, a much needed reformation of the law has at last been effected. Hitherto the purchase of a reversion was liable to be set aside by the Court of Chancery at any time, on the ground that the full value had not been paid by the purchaser; and in that case, the purchaser received back only the amount of the purchase money, with *simple interest* at 5 per cent for the time which had elapsed since the purchase. No good reason can be given why the purchase of a reversion should not be subject to the same regulations as apply to other kinds of property; and this will, in future, be the case. Although the first effect of the Act is to legalize the sale of reversions at a lower price than has been hitherto allowed, yet it is probable that by increasing the

competition among purchasers, the Act will eventually raise the average selling value of reversions.

ANNO TRICESIMO PRIMO VICTORIÆ REGINÆ.

CAP. IV.

An Act to amend the Law relating to Sales of Reversions.
[*7th December*, 1867.]

WHEREAS it is expedient to amend the Law, as administered in Courts of Equity, with respect to Sales of Reversions:

Be it enacted by the Queen's most Excellent Majesty, by and with the Advice and Consent of the Lords Spiritual and Temporal, and Commons, in this present Parliament assembled, and by the Authority of the same, as follows:

No Purchase made bonâ fide, of Reversionary Interests to be set aside merely on the Ground of Undervalue.

1. No Purchase, made *bonâ fide* and without Fraud or unfair Dealing, of any Reversionary Interest in Real or Personal Estate shall hereafter be opened or set aside merely on the Ground of Undervalue.

Interpretation of "Purchase."

2. The Word "Purchase" in this Act shall include every Kind of Contract, Conveyance, or Assignment under or by which any beneficial Interest in any kind of Property may be acquired.

Commencement of Act.

3. This Act shall come into operation on the First Day of *January*, One thousand eight hundred and sixty-eight, and shall not apply to any Purchase concerning which any Suit shall be then depending.

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*A Budget of Paradoxes.** *By* PROFESSOR DE MORGAN.

(Continued from vol. xiii., page 245.)

No. XXI. 1854—1855.

Calcolo decidozzinale del Barone Silvio Ferrari. Turin, 1854, 4to.

This is a serious proposal to alter our numeral system and to count by twelves. Thus 10 would be twelve, 11 thirteen, &c., two new symbols being invented for ten and eleven. The names of numbers must of course be changed. There are persons who think such changes practicable. I thought this proposal absurd when I first saw it, and I think so still: but the one I shall presently describe beats it so completely in that point, that I have not a smile left for this one.

* In continuing, with the permission of the author, this reprint of the Budget of Paradoxes, we have omitted certain articles, which we thought would have little or no interest for our readers. Mr. De Morgan (without objecting to such omissions) wishes it to be clearly explained that he is in no way answerable for them, or for any others that may occur. He desires to avoid the possibility of any change or suspension of his opinion being inferred. ED. *J. I. A.*

The successful and therefore probably true theory of Comets. London, 1854. (4 pp. duodecimo.)

The author is the late Mr. Peter Legh, of Norbury Booths Hall, Knutsford, who published for eight or ten years the *Ombrological Almanac,* a work of asserted discovery in meteorology. The theory of comets is that the joint attraction of the new moon and several planets in the direction of the sun, draws off the gases from the earth, and forms these cometic meteors. But how these meteors come to describe orbits round the sun, and to become capable of having their returns predicted, is not explained.

The Mormon, New York, Saturday, Oct. 27, 1855.

A newspaper, headed by a grand picture of starred and striped banners, beehive, and eagle surmounting it. A scroll on each side: on the left, "Mormon creed. Mind your own business. Brigham Young": on the right, "Given by inspiration of God. Joseph Smith." A leading article on the discoveries of Prof. Orson Pratt says "Mormonism has long taken the lead in religion: it will soon be in the van both in science and politics." At the beginning of the paper is Prof. Pratt's "Law of Planetary Rotation." The cube roots of the densities of the planets are as the square roots of their periods of rotation. The squares of the cube roots of the masses divided by the squares of the diameters are as the periods of rotation. Arithmetical verification attempted, and the whole very modestly stated and commented on. Dated G. S. L. City, Utah Ter., Aug. 1, 1855. If the creed, as above, be correctly given, no wonder the Mormonites are in such bad odour.

The two estates; or both worlds mathematically considered. London, 1855, small (pp. 16).

The author has published mathematical works with his name. The present tract is intended to illustrate mathematically a point which may be guessed from the title. But the symbols do very little in the way of illustration: thus, x being the *present value* of the future estate (eternal happiness), and a of all that this world can give, the author impresses it on the mathematician that, x being infinitely greater than a, $x+a=x$, so that a need not be considered. This will not act much more powerfully on a mathematician by virtue of the symbols than if those same symbols had been dispensed with: even though, as the author adds, "It was this method of neglecting infinitely small quantities that Sir Isaac Newton was indebted to for his greatest discoveries."

There has been a moderate quantity of well-meant attempt to enforce, sometimes motive, sometimes doctrine, by arguments drawn from mathematics, the proponents being persons unskilled in that science for the most part. The ground is very dangerous: for the illustration often turns the other way with greater power, in a manner which requires only a little more knowledge to see. I have, in my life, heard from the pulpit or read, at least a dozen times, that all sin is infinitely great, proved as follows. The greater the being, the greater the sin of any offence against him: therefore the offence committed against an infinite being is infinitely great. Now the mathematician, of which the proposers of this argument are not aware, is perfectly familiar with quantities which increase together, and never cease increasing, but so that one of them remains finite when the other becomes infinite. In fact the argument is a perfect *non sequitur*. Those who propose it have in their minds, though in a cloudy and indefinite form, the idea of the increase of guilt being *proportionate* to the increase of greatness in the being offended. But this it would never do to state: for by such statement not only would the argument lose all that it has of the picturesque, but the asserted premise would have no strong air of exact truth. How could any one undertake to appeal to conscience to declare that an offence against a being $4\frac{7}{10}$ times as great as another is exactly, no more and no less, $4\frac{7}{10}$ times as great as an offence against the other?

My old friend, the late Dr. Olinthus Gregory, who was a sound and learned mathematician, adopted this dangerous kind of illuslustration in his Letters on the Christian Religion. He argued, by parallel, from what he supposed to be the necessarily mysterious nature of the *impossible* quantity of algebra to the necessarily mysterious nature of certain doctrines of his system of Christianity. But all the difficulty and mystery of the impossible quantity is now cleared away by the advance of algebraical thought: and yet Dr. Gregory's book continues to be sold, and no doubt the illustration is still accepted as appropriate.

The mode of argument used by the author of the tract above named has a striking defect. He talks of reducing this world and the next to "present value," as an actuary does with successive lives or next presentations. Does value make interest? and if not, why? And if it do, then the present value of an eternity is *not* infinitely great. Who is ignorant that a perpetual annuity at 5 per cent. is worth only twenty years' purchase? This point ought to be discussed by a person who treats heaven as a deferred

perpetual annuity. I do not ask him to do so, and would rather he did not; but if he *will* do it, he must either deal with the question of discount, or be asked the reason why.

When a very young man, I was frequently exhorted to one or another view of religion by pastors and others who thought that a mathematical argument would be irresistible. And I heard the following more than once, and have since seen it in print, I forget where. [It is Pascal's, as hereafter noted]. Since eternal happiness belonged to the particular views in question, a benefit infinitely great, then even if the probability of their arguments were small, or even infinitely small, yet the product of the chance and benefit, according to the usual rule, might give a result which no one ought in prudence to pass over. They did not see that this applied to all systems as well as their own. I take this argument to be the most perverse of all the perversions I have heard or read on the subject: there is some high authority for it, whom I forget.

The moral of all this is, that such things as the preceding should be kept out of the way of those who are not mathematicians, because they do not understand the argument; and of those who are, because they do.

No. XXII. 1855.

The Sentinel, vol ix. no. 27. London, Saturday, May 26, 1855.

This is the first London number of an Irish paper, Protestant in politics. It opens with 'Suggestions on the subject of a *Novum Organum Moralium*,' which is the application of algebra and the differential calculus to morals, socials, and politics. There is also a leading article on the subject, and some applications in notes to other articles. A separate publication was afterwards made with the addition of a long Preface; the author being a clergyman who I presume must have been the editor of the *Sentinel*.

Suggestions as to the employment of a *Novum Organum Moralium*. Or, thoughts on the nature of the Differential Calculus, and on the application of its principles to metaphysics, with a view to the attainment of demonstration and certainty in moral, political and ecclesiastical affairs. By Tresham Dames Gregg, Chaplain of St. Mary's, within the church of St. Nicholas intra muros, Dublin. London, 1859, 8vo. (pp. xl + 32).

I have a personal interest in this system, as will appear from the following extract from the newspaper:—

"We were subsequently referred to De Morgan's 'Formal Logic' and Boole's 'Laws of Thought,' both very elaborate works, and greatly in the direction taken by ourselves. That the writers amazingly surpass us in learning we most willingly admit, but we venture to pronounce of both their learned treatises, that they deal with the subject in a mode that is scholastic to an excess.... That their works have been for a considerable space of time before the world and effected nothing, would argue that they have overlooked the vital nature of the theme.... On the whole, the writings of De Morgan and Boole go to the full justification of our principle without in any wise so trenching upon our ground as to render us open to reproach in claiming our Calculus as a great discovery.... But we renounce any paltry jealousy as to a matter so vast. If De Morgan and Boole have had a priority in the case, to them we cheerfully shall resign the glory and honour. If such be the truth, they have neither done justice to the discovery, nor to themselves [quite true]. They have, under the circumstances, acted like 'the foolish man, who roasteth not that which he taketh in hunting.'.... It will be sufficient for us, however, to be the Columbus of these great Americi, and popularise what they found, *if* they found it. We, as from the mountain top, will then become *their* trumpeters, and cry glory to De Morgan and glory to Boole, under him who is the source of all glory, the only good and wise, to whom be glory for ever! *If* they be our predecessors in this matter, they have, under Him, taken moral questions out of the category of probabilities, and rendered them perfectly certain. In that case, let their books be read by those who may doubt the principles this day laid before the world as a great discovery, by our newspaper. Our cry shall be *ευρηκασι*! Let us hope that they will join us, and henceforth keep their right [*sic*] from under their bushel."

For myself, and for my old friend Mr. Boole, who I am sure would join me, I disclaim both priority, simultaneity, and posteriority, and request that nothing may be trumpeted from the mountain top except our abjuration of all community of thought or operation with this *Novum Organum.*

To such community we can make no more claim than Americus could make to being the forerunner of Columbus who popularized his discoveries. We do not wish for any *ευρηκασι*, and not even for *ἑυρηκασι*. For self and Boole, I point out what would have convinced either of us that this house is divided against itself.

A being the apostolic element, δ the doctrinal element, and X the body of the faithful, the church is AδX, we are told. Also, that if A become negative, or the Apostolicity become Diabolicity [my words]; or if δ become negative, and doctrine become heresy; or if X become negative, that is, if the faithful become unfaithful; the church becomes negative, "the very opposite of what it ought to be." For self and Boole, I admit this. But—which is not noticed—if A and δ should *both* become negative, diabolical origin and heretical doctrine, then the church, AδX, is still positive, what

it ought to be, unless X be also negative, or the people unfaithful to it, in which case it is a bad church. Now self and Boole—though I admit I have not asked my partner—are of opinion that a diabolical church with false doctrine does harm when the people are faithful, and can do good only when the people are unfaithful. We may be wrong, but this is what we *do* think. Accordingly, we have caught nothing, and can therefore roast nothing of our own: I content myself with roasting a joint of Mr. Gregg's larder.

These mathematical vagaries have uses which will justify a large amount of quotation: and in a score of years this may perhaps be the only attainable record. I therefore proceed.

After observing that by this calculus juries (heaven help them! say I) can calculate damages "almost to a nicety," and further that it is made abundantly evident that cex is "the general expression for an individual," it is noted that the number of the Beast is not given in the Revelation in words at length, but as χξϛ′. On this the following remark is made:—

> "Can it be possible that we have in this case a specimen given to us of the arithmetic of heaven, and an expression revealed, which indicates by its function of addibility, the name of the church in question, and of each member of it; and by its function of multiplicability the doctrine, the mission, and the members of the great Synagogue of Apostacy? We merely propound these questions;—we do not pretend to solve them."

After a translation in blank verse—a very pretty one—of the 18th Psalm, the author proceeds as follows, to render it into differential calculus:—

> "And the whole tells us just this, that David did what he could. He augmented those elements of his constitution which were (*exceptis excipiendis*) subject to himself, and the Almighty then augmented his personal qualities, and his vocational *status*. Otherwise, to throw the matter into the expression of our notation, the variable e was augmented, and cx rose proportionally. The law of the variation, according to our theory, would be thus expressed. The resultant was David the king cex [$c=r$?] (who had been David the shepherd boy), and from the conditions of the theorem we have
>
> $$\frac{du}{de}=ce\frac{dx}{de}+ex\frac{dc}{de}+cx$$
>
> which, in the terms of ordinary language, just means, that the increase of David's educational excellence or qualities—his piety, his prayerfulness, his humility, obedience, &c. was so great, that when multiplied by his original talent and position, it produced a product so great as to be equal in its amount to royalty, honour, wealth, and power, &c.: in short, to all the attributes of majesty."

The "solution of the family problem" is of high interest. It is to determine the effect on the family in general from a change [of conduct] in one of them. The person chosen is one of the maid-servants.

"Let cex be the father; $c_1 e_1 x_1$ the mother, &c. The family then consists of the maid's master, her mistress, her young master, her young mistress, and fellow servant. Now the master's calling (or c) is to exercise his share of control over this servant, and mind the rest of his business call this remainder a, and let his calling generally, or all his affairs, be to his maid-servant as $m : y$, *i.e.*, $y = \frac{mz}{c}$; . . . . and this expression will represent his relation to the servant. Consequently,

$$cex = \left(a + \frac{mz}{c}\right) ex; \text{ otherwise } \left(a + \frac{mz}{c}\right) ex$$

is the expression for the father when viewed as the girl's master."

I have no objection to repeat so far; but I will not give the formula for the maid's relation to her young master; for I am not quite sure that all young masters are to be trusted with it. Suffice it that the son will be affected directly as his influence over her, and inversely as his vocational power: if then he should have some influence and no vocational power, the effect on him would be infinite. This is dismal to think of. Further, the formula brings out that if one servant improve, the other must deteriorate, and *vice versâ*. This is not the experience of most families: and the author remarks as follows:—

"This is, we should venture to say, a very beautiful result, and we may say it yielded us no little astonishment. What our calculation might lead to we never dreamt of; that it should educe a conclusion so recondite that our unassisted power never could have attained to, and which, if we could have conjectured it, would have been at best the most distant probability, that conclusion being itself, as it would appear, the quintessence of truth, afforded us a measure of satisfaction that was not slight."

That the writings of Mr. Boole and myself "go to the full justification of" this "principle," is only true in the sense in which the Scotch use, or did use, the word *justification.*

No. XXIII. 1856.

The last number of this budget had stood in type for months, waiting until there should be a little cessation of correspondence more connected with the things of the day. I had quite forgotten what it was to contain; and had little thought, when I read the proof,

that my allusions to my friend Mr. Boole, then in life and health, would not be printed until many weeks after his death. Had I remembered what my last number contained, I should have added my expression of regret and admiration to the numerous obituary testimonials which this great loss to science has called forth.

The system of logic alluded to in the last number of this series is but one of many proofs of genius and patience combined. I might legitimately have entered it among my *paradoxes*, or things counter to general opinion: but it is a paradox which, like that of Copernicus, excited admiration from its first appearance. That the symbolic processes of algebra, invented as tools of numerical calculation, should be competent to express every act of thought, and to furnish the grammar and dictionary of an all-containing system of logic, would not have been believed until it was proved. When Hobbes, in the time of the Commonwealth, published his 'Computation or Logique,' he had a remote glimpse of some of the points which are placed in the light of day by Mr. Boole. The unity of the forms of thought in all the applications of reason, however remotely separated, will one day be matter of notoriety and common wonder: and Boole's name will be remembered in connexion with one of the most important steps towards the attainment of this knowledge.

The Decimal System as a whole. By Dover Statter, London and Liverpool, 1856, 8vo.

The proposition is to make everything decimal. The day, now 24 hours, is to be made 10 hours. The year is to have ten months, Unusber, Duober, &c. Fortunately there are ten commandments, so there will be neither addition to, nor deduction from, the moral law. But the twelve apostles! Even rejecting Judas, there is a whole apostle of difficulty. These points the author does not touch.

The first Book of Phonetic Reading. London. Fred. Pitman, Phonetic Depot, 20, Paternoster Row, 1856, 12mo.

The Phonetic Journal. Devoted to the propagation of phonetic reading, phonetic longhand, phonetic shorthand, and phonetic printing. No. 46. Saturday, 15 November, 1856. Vol. 15.

I write the titles of a couple out of several tracts which I have by me. But the number of publications issued by the promoters of this spirited attempt is very large indeed. The attempt itself has had no success with the mass of the public. This I do not regret. Had the world found that the change was useful, I should

have gone contentedly with the stream; but not without regretting our old language. I admit the difficulties which our unpronunciable spelling puts in the way of learning to read: and I have no doubt that, as affirmed, it is easier to teach children phonetically, and afterwards to introduce them to our common system, than to proceed in the usual way. But by the usual way I mean proceeding by letters from the very beginning. If, which I am sure is a better plan, children be taught at the commencement very much by *complete* words*, as if they were learning Chinese, and be gradually accustomed to resolve the known words into letters, a fraction, perhaps a considerable one, of the advantage of the phonetic system is destroyed. It must be remembered that a phonetic system can only be an approximation. The differences of pronunciation existing among educated persons are so great, that, on the phonetic system, different persons ought to spell differently.

But the phonetic party have produced something which will immortalize their plan: I mean their *short hand,* which has had a fraction of the success it deserves. All who know anything of shorthand must see that nothing but a phonetic system can be worthy of the name: and the system promulgated is skilfully done. Were I a young man I should apply myself to it systematically. I believe this is the only system in which books were ever published. I wish some one would contribute to a public journal a brief account of the dates and circumstances of the phonetic movement, not forgetting a list of the books published in shorthand.

A child beginning to read by himself, may owe terrible dreams and waking images of horror to our spelling; as I did when six years old. In one of the common poetry-books there is an admonition against confining little birds in cages, and the child is asked what if a great giant, amazingly strong, were to take you away, shut you up,

And feed you with vic-tu-als you ne-ver could bear.

The book was hyphened for the beginner's use; and I had not the least idea that *vic-tu-als* were *vittles:* by the sound of the word I judged they must be of iron; and it entered into my soul.

The worst of the phonetic shorthand books is that they nowhere, so far as I have seen, give *all* the symbols, in every stage of advancement, together, in one or following pages. It is symbols

* The Robinson Crusoe in words of one syllable (published August, 1867) would be an excellent reading book as a sequence to 'The dog bit the cat,' &c.

and talk, more symbols and more talk, &c. A universal view of the signs ought to begin the works.

Ombrological Almanac. Seventeenth year. An essay on Anemology and Ombrology. By Peter Legh, Esq. London, 1856, 12mo.

Mr. Legh, already mentioned, was an intelligent country gentleman, and a legitimate speculator. But the clue was not reserved for him.

The proof that the three angles of a triangle are equal to two right angles looked for in the inflation of the circle. By Gen. Perronet Thompson. London, 1856, 8vo. (pp. 4.)

Another attempt, the third, at this old difficulty, which cannot be put into few words of explanation.

Comets considered as volcanoes, and the cause of their velocity and other phenomena thereby explained. London (*circa* 1856), 8vo.

The title explains the book better than the book explains the title.

1856. A stranger applied to me to know what the ideas of a friend of his were worth upon the magnitude of the earth. The matter being one involving points of antiquity, I mentioned various persons whose speculations he seemed to have ignored; among others, Thales. The reply was, "I am instructed by the author to inform you that he is perfectly acquainted with the works of Thales, Euclid, Archimedes,......" I had some thought of asking whether he had used the Elzevir edition of Thales, which is known to be very incomplete, or that of Prof. Niemand, with the lections, Nirgend, 1824, 2 vols. folio; just to see whether the last would not have been the very edition he had read. But I refrained, in mercy.

The moon is the image of the Earth, and is not a solid body. By The Longitude. (Private Circulation.) In five parts. London, 1856, 1857, 1857; Calcutta, 1858, 1858, 8vo.

The earth is "brought to a focus"; it describes a "looped" orbit round the sun. The eclipse of the sun is thus explained: "At the time of eclipses, the image is more or less so directly before or behind the earth that, in the case of new moon, bright rays of the sun fall and bear upon the spot where the figure of the earth is brought to a focus, that is, bear upon the image of the earth, when a darkness beyond is produced reaching to the earth, and the sun becomes more or less eclipsed." How the earth is "brought to a focus" we do not find stated. Writers of this

kind always have the argument that some things which have been ridiculed at first have been finally established. Those who put into the lottery had the same kind of argument; but were always answered by being reminded how many blanks there were to one prize. I am loath to pronounce against anything: but it does force itself upon me that the author of these tracts has drawn a blank.

No. XXIV. 1856—1858.

Times, April 6 or 7, 1856. The moon has no rotary motion.

A letter from Mr. Jellinger Symons, inspector of schools, which commenced a controversy of many letters and pamphlets. This dispute comes on at intervals, and will continue to do so. It sometimes arises from inability to understand the character of simple rotation, geometrically; sometimes from not understanding the mechanical doctrine of rotation.

> Lunar Motion. The whole argument stated, and illustrated by diagrams; with letters from the Astronomer Royal. By Jellinger C. Symons. London, 1856, 8vo.

The Astronomer Royal endeavoured to disentangle Mr. J. C. Symons, but failed. Mr. Airy can correct the error of a ship's compasses, because he can put her head which way he pleases: but this he cannot do with a speculator.

> The Doctrine of the Moon's Rotation, considered in a letter to the Astronomical Censor of the *Athenæum*. By Jones L. Mac-Elshender. Edinburgh, 1856, 8vo.

This is an appeal to those cultivated persons who will read it "to overrule the *dicta* of judges who would sacrifice truth and justice to professional rule, or personal pique, pride, or prejudice"; meaning, the great mass of those who have studied the subject. But how? Suppose the "cultivated persons" were to side with the author, would those who have conclusions to draw and applications to make consent to be wrong because the "general body of intelligent men," who make no special study of the subject, are against them? They would do no such thing: they would request the general body of intelligent men to find their own astronomy, and welcome. But the truth is that this intelligent body knows better: and no persons know better that they know better than the speculators themselves.

But suppose the general body were to combine, in opposition to those who have studied. Of course all my list must be admitted

to their trial; and then arises the question whether both sides are to be heard. If so, the general body of the intelligent must hear all the established side have to say: that is, they must become just as much of students as the inculpated orthodox themselves. And will they not then get into *professional rule*, pique, pride, and prejudice, as the others did? But if, which I suspect, they are intended to judge just as they are, they will be in a rare difficulty. All the paradoxers are of like pretensions: they cannot, as a class, be right, for each one contradicts a great many of the rest. There will be the puzzle which silenced the crew of the cutter in Marryat's novel of the Dog-fiend. "A tog is a tog," said Jansen.—"Yes," replied another, "we all know a dog is a dog; but the question is —Is *this* dog a dog?" And this question would arise upon every dog of them all.

Zetetic Astronomy: Earth not a globe. 1857 (broadsheet).

Though only a travelling lecturer's advertisement, there are so many arguments and quotations that it is a little pamphlet. The lecturer gained great praise from provincial newspapers for his ingenuity in proving that the earth is a flat, surrounded by ice. Some of the journals rather incline to the view: but the *Leicester Advertiser* thinks that the statements "would seem very seriously to invalidate some of the most important conclusions of modern astronomy," while the *Norfolk Herald* is clear that "there must be a great error on one side or the other." This broadsheet is printed at Aylesbury in 1857, and the lecturer calls himself *Parallax:* but at Trowbridge, in 1849, he was S. Goulden. In this last advertisement is the following announcement—"A paper on the above subjects was read before the Council and Members of the Royal Astronomical Society, Somerset House, Strand, London (Sir John F. W. Herschel, President), Friday, Dec. 8, 1848." No account of such a paper appears in the *notice* for that month: I suspect that the above is Mr. S. Goulden's way of representing the following occurrence. Dec. 8, 1848, the Secretary of the Astronomical Society said, at the close of the proceedings,—"Now, gentlemen, if you will promise not to tell the Council, I will read something for your amusement": and he then read a few of the arguments which had been transmitted by the lecturer. The fact is worth noting that from 1849 to 1857, arguments on the roundness or flatness of the earth did itinerate. I have no doubt they did much good: for very few persons have any distinct idea of the evidence for the rotundity of the earth. The *Blackburn*

Standard and *Preston Guardian* (Dec. 12 and 16, 1849) unite in stating that the lecturer ran away from his second lecture at Burnley, having been rather too hard pressed at the end of his first lecture to explain why the large hull of a ship disappeared before the sails. The persons present and waiting for the second lecture assuaged their disappointment by concluding that the lecturer had slipped off the icy edge of his flat disk, and that he would not be seen again till he peeped up on the opposite side.

But, strange as it may appear, the opposer of the earth's roundness has more of a case—or less of a want of case—than the arithmetical squarer of the circle. The evidence that the earth is round is but cumulative and circumstantial: scores of phenomena ask, separately and independently, what other explanation can be imagined except the sphericity of the earth. The evidence for the earth's figure is tremendously powerful of its kind; but the proof that the circumference is 3·14159265 . . . times the diameter is of a higher kind, being absolute mathematical demonstration.

No. XXV. 1859.

The great Pyramid. Why was it built? And who built it? By John Taylor, 1859, 12mo.

This work is very learned, and may be referred to for the history of previous speculations. It professes to connect the dimensions of the pyramid with a system of metrology which is supposed to have left strong traces in the systems of modern times; showing the Egyptians to have had good approximate knowledge of the dimensions of the earth, and of the quadrature of the circle. These are points on which coincidence is hard to distinguish from intention. Sir John Herschel noticed this work, and gave several coincidences, in the *Athenæum*, Nos. 1696 and 1697, April 28 and May 5, 1860: and there are some remarks by Mr. Taylor in No. 1701, June 2, 1860.

Mr. Taylor's most recent publication is

The battle of the standards: the ancient, of four thousand years, against the modern, of the last fifty years—the less perfect of the two. London, 1864, 12mo.

This is intended as an appendix to the work on the pyramid. Mr. Taylor distinctly attributes the original system to revelation, of which he says the Great Pyramid is the record. We are advancing, he remarks, towards the end of the Christian Dispen-

sation, and he adds that it is satisfactory to see that we retain the standards which were given by unwritten revelation 700 years before Moses. This is lighting the candle at both ends; for myself, I shall not undertake to deny or affirm either what is said about the dark past or what is hinted about the dark future.

My old friend Mr. Taylor is well known as the author of the argument which has convinced many, even most, that Sir Philip Francis was Junius: pamphlet, 1813; supplement, 1817; second edition 'The Identity of Junius with a distinguished living character established,' London, 1818, 8vo. Sir Philip Francis, in a short conversation with him, made only this remark, "You may depend upon it you are quite mistaken": the phrase appears to me remarkable; it has an air of criticism on the book, free from all personal denial. [This from Mr. Taylor himself.] I have heard, but not from any such degree of nearness to the source, though not remotely, [it was repeated to Mr. Taylor by the person who heard it] that Sir Philip said, speaking of writers on the question,—"Those fellows, for half-a-crown, would prove that Jesus Christ was Junius."

Mr. Taylor implies, I think, that he is the first who started the suggestion that Sir Philip Francis was Junius, which I have no means either of confirming or refuting. If it be so [and I now know that Mr. Taylor himself never heard of any predecessor], the circumstance is very remarkable: it is seldom indeed that the first proposer of any solution of a great and vexed question is the person who so nearly establishes his point in general opinion as Mr. Taylor has done.

As to the Junius question in general, there is a little bit of the philosophy of horse-racing which may be usefully applied. A man who is so confident of his horse that he places him far above any other, may nevertheless, and does, refuse to give odds against all the field: for many small adverse chances united make a big chance for one or other of the opponents. I suspect Mr. Taylor has made it at least 20 to 1 for Francis against any one competitor who has been named: but what the odds may be against the whole field is more difficult to settle. What if the real Junius should be some person not yet named?

Mr. Jopling, *Leisure Hour*, May 23, 1863, relies on the porphyry coffer of the great pyramid, in which he finds "the most ancient and accurate standard of measure in existence."

I am shocked at being obliged to place a thoughtful and learned writer, and an old friend, before such a successor as he here meets

with. But chronological arrangement defies all other arrangement.

I had hoped that the preceding account would have met Mr. Taylor's eye in print: but he died during the last summer. For a man of a very thoughtful and quiet temperament, he had a curious turn for vexed questions. But he reflected very long and very patiently before he published: and all his works are valuable for their accurate learning, whichever side the reader may take.

HOME AND FOREIGN INTELLIGENCE.

THE SCOTTISH WIDOWS' FUND LIFE ASSURANCE SOCIETY.

Founded A.D. 1815.

SEVENTH DIVISION OF PROFITS.

REPORT BY THE MANAGER ON THE INVESTIGATION OF THE SOCIETY'S AFFAIRS, MADE AS AT 31ST DECEMBER 1866.

THE operations of the seven years ending 31st December last have been marked by several important features which it seems desirable to bring to the recollection of the Court of Directors, at the outset of the present Report.

Change of Laws.

The Court is aware that the Society was originally constituted on the basis of the Northampton Table of Mortality and on the assumption of an improvement of money at the rate of 4 per cent. At each periodical ascertainment of Surplus it was required that there should be reserved from division at least one-third of the Surplus, and also that, from the remaining two-thirds, there should be set aside a sum equal to the value of the Intermediate Bonus, payable on such Policies as should become Claims before the next period of Division. The six Investigations, prior to the present, were made according to these Rules. To persons not intimately acquainted with the subject, the reserve of the large sums referred to, out of what the system represented to be actual Surplus, naturally suggested the belief that there was a tendency in the original constitution to the accumulation of undivided profit, in which the Members whose payments had created it might never participate. On the other hand, it was shown, by *approximate* Valuations made in 1852 and 1859 according to the Carlisle Table of Mortality, and 3 per cent interest, that the tendency really was in the opposite direction. The probabilities deduced from the Assumed Mortality of the Northampton Table were evidently so wide of the truth, and the whole results brought out by the use of them (even with the compensatory reserves referred to) were seen to be so doubtful and misleading, that it had become absolutely necessary to abandon the original basis of Valuation, and to adopt another and more reliable one in its stead. The new basis of the Carlisle 3 per cent Table was, in the end of 1864, finally adopted.

The present Investigation has therefore been made, for the first time in the history of the Society, according to these new and approved data of Valuation, in the exact and detailed manner to be afterwards explained.

Increase of Business.

A remarkable increase in the Business of the Society has taken place during the course of the Septennial period now to be reported upon. The New Assurances effected in each of the seven years, and the new Premiums payable thereon, have been as in the following

VIDIMUS OF NEW BUSINESS TRANSACTED.

Year of Issue.	Policies issued.	Amount Assured.	First Premiums thereon.
During Year 1860	600	£380,305 0 0	£12,530 6 1
Do. Year 1861	603	374,599 0 0	13,438 5 2
Do. Year 1862	981	666,834 18 5	23,251 18 5
Do. Year 1863	1169	882,485 13 3	30,006 0 7
Do. Year 1864	1263	876,349 13 6	30,128 17 4
Do. Year 1865	1603	1,045,497 13 0	35,106 17 9
Do. Year 1866	1893	1,235,812 9 7	40,934 5 4
Total.	**8112**	**£5,461,884 7 9**	**£185,396 10 8**

These figures do not represent any business re-assured with other Offices.

Interest realised.

In 1860 the Average Rate of Interest realised was			£4	1	3	per cent.
In 1861	"	"	4	2	5	"
In 1862	"	"	4	3	0	"
In 1863	"	"	4	2	10	"
In 1864	"	"	4	4	3	"
In 1865	"	"	4	7	0	"
In 1866	"	"	4	10	6	"

The average rate prior to the commencement of this Septennium was about 4 per cent.

High rates of Interest have prevailed of late years upon the best securities, and new channels of Investment, well adapted for the purposes of the Society, have been opened up, such as Rent-charges on landed estates, and Rates leviable by Statute from real property in Towns and Rural Parishes.

Expenses of Management.

The whole Expenses of Management at the Head Office and Agencies, which include Commission and charges of every kind, have, during the seven years, been 6¼ per cent of the Revenue. During the Septennium a large part of the old Business, on which little or no commission was payable, has lapsed by death of the lives assured, and this having been replaced by large introductions of New Assurances from Agencies, the *ratio* of expenditure in this respect has, of course, considerably increased, as shown by the fact that the rate of the whole expenses during the previous Septennium was only 4¼ per cent of the revenue.

It is important to observe however that, viewing the total expenses of management in relation to the New Business annually introduced to the Society, the *ratio* which the total expenditure bears to the first Premiums of New Assurances issued is, at the close of the Septennium, only one-half what it was at its commencement.

Increase of Society's Resources.

The Funds and Revenue of the Society have increased during the Seven years as in the following Abstract.

	Funds.	Annual Revenue.
1860. January 1, commencement of Septennium	£3,518,230 6 9	£412,767 9 2
1860. December 31, end of 1st year	3,601,763 14 7	426,653 5 1
1861. " end of 2d year	3,720,200 4 5	442,438 0 6
1862. " end of 3d year	3,849,127 8 1	458,691 12 1
1863. " end of 4th year	3,975,325 17 10	479,791 10 2
1864. " end of 5th year	4,086,824 17 0	505,424 10 7
1865. " end of 6th year	4,230,405 4 11	537,568 4 10
1866. Close of Septennium	**4,371,995 0 1**	**589,440 9 8**

	Funds.	Annual Revenue.
Increase during the Seven Years	**£853,764 13 4**	**£176,673 0 6**

Mortality Experience.

Comparing the actual mortality results with the expectation according to the Carlisle Table, the facts, both with reference to the number of deaths and amount of claims, are shown as follow:—

Year.	Deaths.	Expectation.	Claims.	Expectation.
During Year 1860	176	183	£277,901	£265,939
" 1861	157	189	265,605	271,858
" 1862	172	197	279,235	284,246
" 1863	206	208	296,427	295,574
" 1864	230	217	328,419	311,376
" 1865	212	231	331,363	326,583
" 1866	245	245	386,511	346,544
Total.	1398	1470	£2,165,461	£2,102,120

Of this amount, (£2,165,461,) £591,538 were Bonus Additions, being £38¼ per cent of the original sums assured under Policies entitled to Bonus.

It appears that, while the number of deaths during the Septennium has been within the limits of the Carlisle Expectations, the amount of Claims payable in consequence thereof has exceeded the expectations. A similar disparity of result is becoming more or less prominent in the experience of Life Assurance Institutions generally, and appears to arise from the method of dividing the Surplus, not in present cash, but by granting what are equivalent to New Assurances in the form of Reversionary Bonus Additions, which, as the Court are aware, are declared upon the Policies without any

reference to the continued health and assurable condition of the Members at the time. This mode of dividing Surplus has obtained large acceptance, and is very generally adopted by Life Assurance Companies.

Ascertainment of Surplus.

Every Policy and Obligation by the Society in existence at 31st December has been made the subject of separate and independent Valuation. These Valuations have been checked at different times by different computers, and the entire Investigation has been tested by another and altogether different method of Valuation, and found to agree therewith, so that perfect reliance may be placed upon the accuracy of the results. It is scarcely necessary to add that the additions to the nett or pure Premiums, technically termed "Loading," have been carefully separated and wholly set aside in valuing the future Premiums. By assuming nett Premiums alone as an Asset, the possibility of anticipating future profits to any extent whatever has been effectually excluded, and all sources of future surplus are thereby preserved intact.

It will be remembered that approximate valuations made at the end of last Septennium showed that the undivided Surplus composed of the Guarantee Fund and other Reserves, amounting to £454,000, as brought out by the Northampton system was to the extent at least of three-fifths of its amount illusory.

It began early in the course of the present Investigation to become manifest that the approximate Valuations of 1852 and 1859 had not fully disclosed the misleading character of the Northampton figures; and when the whole Calculations were brought to a close, it was found that if there was any Balance at all of Realised Profits remaining undivided among the Members at 1859, it could only have been of small amount, probably not more than a few thousand pounds; and that the supposed Surplus of undivided Profit at that time was not only to the extent of three-fifths, but nearly altogether, without existence, and therefore that whatever Surplus arises under the present Investigation must have been realised almost entirely during the seven years since 1st January 1860.

* * * * *

	£	s.	d.
The total Surplus of the Septennium is . .	£834,183	10	1
Whereof there has been paid to the Representatives of Members who died during the seven years	118,395	13	2
Leaving . . .	£715,787	16	11

now to be apportioned among the Members in terms of the Laws.

Comparative amount of Surplus.

	Premiums Paid.	Cash Surplus Divided.	Percentage of Surplus on Premiums Paid.
Septennium ending 1852 ..	£1,442,446	£525,055	36.400 per cent.
Do. 1859 ..	1,778,187	647,512	36.411 per cent.
Do. 1866 ..	2,176,716	834,183	38.323 per cent.

Thus a larger percentage of Premiums is returnable in the form of Additions to the Policies on this occasion than on any previous one.

Second Comparative View.—At 1859 the rate of Reversionary Bonus declared (£1 : 12 : 6 per cent per annum on original sums assured and previous Bonuses) yielded, upon the whole, the largest Reversionary Bonus which had been declared by the Society up to that date. Were the Surplus which has arisen during this last Septennium distributed among the Members in precisely the same way, it would yield the higher rate of £1 : 13s. per cent, but of course upon the still further accumulated amount. This would yield on Policies of from one to seven years' standing a slightly larger Bonus than Policies of like standing received at 1859, and on all Policies issued prior to the year 1854, a larger amount of Bonus by sums varying from £15 to £30 per £1000 assured.

The above rate of £1 : 13s. per cent is equivalent to Reversionary Bonuses on the Premiums paid during the seven years varying from 50 per cent on the youngest Policies, to 140 per cent on the oldest Policies, the average of the whole being £63 per cent.

Division of Surplus among the Policyholders.

Under the new Law of Division which comes into force for the first time at the present Septennial Investigation, the Balance of Surplus, £715,787 : 16 : 11, is now divisible among the Members—partly in the form of Bonus, and partly as a rateable proportion of Guarantee Fund retained at the credit of each Policy in the form of a cash deposit, bearing Interest at 3 per cent from 31st December 1866, and payable in the event of death occurring before another Division.

In order to give effect to this double form of Addition to the Policies, the Extraordinary Court of Directors are required to divide the Surplus into Bonus Fund and Guarantee Fund, and to fix the latter at not less than £5 per cent of the value of the Total Liabilities at the time. There being no grounds on which more than £5 per cent could be considered proper in fixing the amount of the Guarantee Fund, the following will be the Division of the Surplus:—

1. Amount carried to GUARANTEE FUND, being £5 per cent on the value of the Society's whole Liabilities (£207,893 : 17 : 2), which, with an unappropriated sum of £10,459, 4s. 9d. amounts to	£218,353	1	11
2. Amount carried to BONUS FUND for Remainder of Surplus Fund	497,434	15	0
Total . .	£715,787	16	11

It has already been stated that if the entire balance of the Surplus Fund, amounting to £715,787 : 16 : 11, were divided in the form of Bonus, it would yield Reversionary Additions at the rate of £1 : 13s. per cent per annum in the compound form. It follows, therefore, that the two additions now to be made to the Policies, under the different names of "Portion of Guarantee Fund" and Bonus, are together the equivalent of that rate of Bonus. That is to say—

The portion of Guarantee Fund to (£715,787 : 16 : 11 : £218,353 : 1 : 11 :: £1 : 13 : 0)	=£0 10 0 p. ct.
The Bonus to (£715,787 : 16 : 11 : £497,434 : 15 : 0 :: £1 : 13 : 0)	=£1 3 0 p. ct.
Together . .	£1 13 0 p. ct.

The Bonus of £1 : 3s per cent will be calculated as on previous occasions in the compound form upon the Original Sum Assured and all former Bonus Additions in existence at 31st December 1866. The effect of that rate of Bonus as an Addition to the Policies is shown in the Table (below).

The Portion of Guarantee Fund will be calculated as a Cash Addition (yielding Interest at 3 per cent from 31st December 1866) also in the compound form, viz.—on the Total Cash Values not only of the Original Sums Assured and Additions at that date, but also on the value of the Bonus of £1 : 3s. per cent now declared.

The amount of Guarantee Fund added to every Policy cannot be conveniently shown in a Table, because it depends upon the age of the lives assured at the date of Valuation, as well as upon the duration of the Policy at that time; the general effect of the division of the Fund among the Members is shown in the Table at the end—the examples being based on the assumption of the lives assured having been 35 years of age at entry. The portion of Guarantee Fund to be added to the Policies will, of course, be greater or less than the sums stated in the Table, according as the age at the time of entry was above or below 35.

There was also declared a Contingent Prospective Bonus on Policies which have become Claims subsequent to 31st December 1866, or may do so on or before 31st December 1873, either by death of the Lives Assured, or by the occurrence of the events on which the sums assured become payable, at a rate one-quarter or 5s. per cent per annum less than the rate of vested Reversionary Bonus above mentioned, on the original amount assured and previous vested Bonuses, including that now declared, remaining in existence at date of claim.

And the said whole Additions are made in the same manner to Policies obtained by single payments, or other mode of contribution different from Annual Payments.

Balance-Sheet of the Scottish Widows' Fund and Life Assurance Society

AS AT 31ST DECEMBER 1866.

LIABILITIES.

I. DEBTS DUE BY THE SOCIETY—

1. Claims by Death of Lives Assured, not yet payable . .	£219,458 16 9
2. Arrears of Annuities unclaimed	352 8 11
3. Commission due to Agents on Premiums in course of collection, Tradesmen's and all other outstanding Accounts at 31st December	8,166 5 8
Carried forward	£227,977 11 4

Brought forward £227,977 11 4

II. Value of the Society's Liabilities under Assurance and Annuity Policies, per Abstract of Valuations—

	Value of Sums Assured and Bonus Additions.			Value of Future Nett Premiums.			Difference, being Nett Liability.		
	£	s.	d.	£	s.	d.	£	s.	d
1.—Assurances.									
Policies with participation . .	7,492,134	19	3	3,954,204	11	0	3,537,930	8	3
Policies without participation .	204,818	9	6	154,995	4	11	49,823	4	7
	7,696,953	8	9	4,109,199	15	11	3,587,753	12	10
Deduct Re-assurances . .	27,520	0	4	23,691	9	9	3,828	10	7
2.—Annuities.	7,669,433	8	5	4,085,508	6	2	3,583,925	2	3
Policies with participation . .	956	4	0	135	0	8	821	3	4
Policies, without participation .	73,613	4	9	2,152	7	7	71,460	17	2
	7,744,002	17	2	4,087,795	14	0	3,656,207	3	2

3,656,207 3 2

Total Liabilities £3,884,184 14 6

Note.—*The gross Annual Premiums payable amount to £376,386 : 12 : 2, of which £79,547 : 0 : 11 per annum is "Loading." The difference of £296,839 : 11 : 3 per annum only is contained in the above valuation.*

Balance—

Being Amount remaining at Credit of the Profit and Loss Account £715,787 16 11

£4,599,972 11 5

ASSETS.

Landed Securities				£2,992,985	19	11
Railway Debentures				398,826	15	8
Redeemable Annuities and Reversions secured over Land and Life Policies				381,410	19	11
Loans to Members of Society on Policies of greater Value . .				530,175	10	11
Government Annuities on Lives of Nominees				21,333	5	0
Government Terminable Annuities				25,098	8	1
House Property and Ground Rents				24,831	13	7
Office Furniture, from which 10 per cent per ann. has been deducted for depreciation				1,388	7	4
Miscellaneous Sums due to the Society—						
Balances due by Agents	£105,740	5	11			
Renewal Premiums, etc., at Head Office, on which the 30 days of grace had not expired	28,604	6	2			
	£134,344	12	1			
Proportion of Interest on Investments from last payment, less Income-Tax . .	42,650	17	11			
Cash in Bank	46,697	4	3			
Cash in Office	228	16	9			
				223,921	11	0

The Nett Fund of the Society as at 31st December is

Assets as below	£4,599,972	11	5
Less debts per Contra	227,977	11	4
NETT FUND .	£4,371,995	0	1

£4,599,972 11 5

Valuation of the Policies.

Description of Transactions.	Original Sums Assured and Bonus Additions at 31st December 1866.			Gross Annual Premiums payable after giving effect to all Reductions and Redemptions.			Nett Annual Premiums.	Nett Value of Liabilities.
I. Assurances.	Col. 1.			Col. 2.			Col. 3.	Col. 12.
1. WITH PARTICIPATION.	£	s.	d.	£	s.	d.	£	£
Whole Life Policies	12,977,747	3	6	355,330	7	1	278,941	3,495,169
Do. definite number of Premiums	82,742	18	1	4,908	3	10	3,757	15,647
Do. do. decreasing Premiums	1,000	0	0	90	10	0	65	331
Do. Premiums decreasing septennially	500	0	0	30	3	9	24	13
Do. Premiums increasing quinquen[ly].	19,900	0	0	579	7	7	411	318
Do. Premiums increasing septennially	2,250	0	0	43	10	5	28	10
Survivorship Policies	500	0	0	12	19	7	9	26
Endowment Assurance Policies	34,375	0	0	1,922	4	11	1,483	1,493
Policies on the Longest Liver	38,854	14	3	407	6	1	355	21,863
Joint Life Policies	26,978	10	3	1,342	14	1	1,003	3,051
Joint Life Endowment Policies	250	0	0	12	13	2	9	9
1. Assurances with Profits	13,185,098	6	1	364,680	0	6	286,085	3,537,930
2. WITHOUT PARTICIPATION.								
Whole Life Policies	289,752	11	4	9,413	0	6	8,635	34,827
Do. definite number of Premiums	500	0	0	56	10	0	52	252
Do. Premiums increasing quinquen[ly].	12,400	0	0	319	1	0	290	224
Do. Premiums increasing septennially	10,499	0	0	342	10	1	311	118
Survivorship Policies	12,165	0	0	190	16	6	160	154
Endowment Assurance Policies	36,400	0	0	2,082	13	3	1,995	9,430
Policies on the Longest Liver	14,299	18	0	360	15	6	315	4,528
Joint Life Policies	700	0	0	41	18	0	38	32
Endowment Policy	100	0	0	..			..	65
Short Term Policies	23,720	0	0	386	17	4	297	193
2. Assurances without Profits	400,536	9	4	13,194	2	2	12,093	49,823
Total Assurances	13,585,634	15	5	377,874	2	8	298,178	3,587,753
Deduct Re-Assurances	58,631	10	0	1,828	6	8	1,615	3,828
Nett Amount of Assurances	13,527,003	5	5	376,045	16	0	296,563	3,583,925
II. Annuities.								
Present and Contingent (*Int.* 3½ *p. cent*)	9,801	7	1	340	16	2	276	72,282
Total of the Results				376,386	12	2	296,839	3,656,207

The extent of the original table as issued by the Society has rendered considerable abridgment necessary; but in further illustration of the mode of valuation we extract the following:—

Whole Life Policies (With Participation).

			£
Column	3.	Nett Annual Premiums	278,941
„	4.	Proportion of Nett Premiums contained in Col. 3 to next Renewal	186,112
„	5.	Annual Loading on Nett Premiums in Col. 3 . . .	76,390
„	6.	Loading on Single Premiums and parts of Premiums redeemed	772
„	7.	Proportion of Loading in Col. 6 to next Renewal . . .	511
„	8.	Value of Sums Assured and Bonuses ($S \times A_{m'}$) . . .	7,368,525
„	9.	Value of paid-up Loading in Col. 6 $L(a_{m'}+1)$ – "Proportion" thereof in Col. 7	8,887
„	10.	Total Value of Liabilities in Cols. 8 and 9	7,377,412
„	11.	Value of Nett Premiums in Col. 3. $p_m(a_{m'}+1)$ – "Proportion" thereof in Col. 4	3,882,243
„	12.	Nett Value of Liabilities, being excess of Col. 10 over Col. 11.	3,495,169
„	13.	New Bonus at £1. 3*s* per cent per annum on Sums Assured and Bonus Additions	835,966
„	14.	Value of New Bonus in Col. 13	493,082
„	15.	Total Value of Policy and Bonuses to 31st December, 1866, being Cols. 12 and 14	3,988,251
„	16.	Portion of Guarantee Fund being £5 per cent on Col. 15 .	199,413

Notation.—S=Sums Assured; A=Value of £1 payable at Death; a=Value of £1 annually; m=Age at Entry; m'=Age at Date of Valuation; L=Loading; p=Nett Premium.

DISTRIBUTION OF THE SURPLUS.

At credit of the Profit and Loss Account . .		£715,787 16 11
Less unappropriated Balance		10,459 4 9
New Bonus	£497,434 15 0	£705,328 12 2
Portion of Guarantee Fund .	207,893 17 2	705,328 12 2

TABLE

SHOWING the ACCUMULATED AMOUNT of POLICIES of £1000.

Year of Entry.	Policy with Vested Additions at 31st Dec. 1866.	Sums Assured and Bonus Additions (exclusive of Guarantee Fund and Interest) payable if Death occur after Payment of the Premium due in the Year. 1867.	1870.	1873.
	£ s. d.	£ s. d.	£ s. d.	£ s. d.
1816 to 1819	2444 3 4	2466 3 3	2532 3 1	2598 3 0
1820	2071 10 6	2090 3 5	2146 2 0	2202 0 8
1830	1808 3 8	1824 9 2	1873 5 7	1922 2 1
1840	1512 17 5	1526 9 9	1567 6 9	1608 3 8
1850	1266 11 7	1277 19 7	1312 3 6	1346 7 5
1860	1080 10 0	1090 4 6	1119 7 11	1148 11 4
1866	1011 10 0	1020 12 1	1047 18 3	1075 4 5

Additions to Policies effected in the Years 1862 to 1866 are not payable if Death occur within Five Years from their respective dates.

EXAMPLES of the PORTION of GUARANTEE FUND, exclusive of Interest thereon, attaching to Policies of £1000, *assuming the age at entry to be* 35.

Year of Entry.	Portion of Guarantee Fund.	Year of Entry.	Portion of Guarantee Fund.
	£ s. d.		£ s. d.
1816	103 6 11	1850	20 4 5
1820	83 6 7	1860	6 11 6
1830	62 15 3	1866	0 17 6
1840	39 10 9		

At each Septennial Division of Profits which the Life assured may survive, the "Portion of Guarantee Fund" will be increased in exact proportion to the then increased value of the Policy—such increased sum to be paid at death, with interest thereon at £3 per cent per annum. On the other hand, if either the Sum Assured or Bonuses be surrendered or reduced, the "Portion of Guarantee Fund" and Interest thereon payable at death will be correspondingly diminished.

Mortality Table for the Seven Years 1860 *to* 1866 *both inclusive.*

Age.	Sum of the annual numbers of lives at risk in each of the seven years.	Number of Deaths.	Percentage of Deaths on the number at risk.
Under 25	2684.166	13	.4843
26 to 30	5395.166	33	.6117
31 " 35	7568.916	53	.7002
36 " 40	9382.250	88	.9379
41 " 45	10326.416	108	1.0458
46 " 50	10368.000	145	1.3985
51 " 55	9450.250	161	1.7036
56 " 60	7352.750	189	2.5704
61 " 65	5136.500	178	3.4654
66 " 70	3126.416	156	4.9897
71 " 75	1742.583	117	6.7142
76 " 80	922.833	94	10.1860
81 " 85	300.500	48	15.9733
86 " 90	58.000	15	25·8620
		1,398	1.8938

Average mean Age at Death 58.0844.

Calculated Number of Lives at Risk.

1860	9,301
1861	9,508
1862	9,759.5
1863	10,324.5
1864	10,802.5
1865	11,522.5
1866	12,596.75
Total . .	73,814.75

THE LONDON LIFE ASSOCIATION.

Established 1806.

REPORT OF THE DIRECTORS TO THE GENERAL COURT, HELD ON THE 26TH JULY, 1865.

When, nearly ten years ago, the Court of Directors proposed to the General Court to close the First Series of Members and commence a Second Series, to be entitled after seven years to a reduction of premium less by 10 per cent. than the First Series, the Directors looked forward to a period when the rate of reduction should have increased so much that it would be to the interest of existing Members to close the Second Series, and to begin a Third.

The time, then uncertain, has now arrived, when, in the opinion of the Directors, the Second Series should be closed,—the reduction to which they are entitled being, for this year, 75 per cent.

During the interval, the prosperity of the Association has been uninterrupted,—the data forming the basis of the Tables from which the Society's calculations are made have been more than realized. It is well known that these data consist of a rate of interest of £4 per cent., and a rate of mortality derived by combining the mortality experienced by the Equitable Assurance Society during 67 years with that of the Government Male Annuitants. More than £4 per cent. on the amount of the Society's funds, invested and uninvested, has constantly been obtained after deduction of Income Tax, and the mortality of the Members has been nearly one-fourth short of that which, from the before-mentioned Table, it was estimated that the Society would experience.

The Directors feel justified in looking forward to a continuance of this state of prosperity. The probability of a realization of an average interest of £4 per cent. appears to the Directors to be at least as great as heretofore, and the probability of a continuation of the low rate of mortality may be inferred from the fact that during the last 35 years the rate has been very nearly the same as during the last ten years. In the longer period, the actual claims by death were only 78¾ per cent. of the estimated claims; and in the shorter period, from 1855, only 77¾ per cent.

Under these circumstances, the Directors are of opinion that the time has come when new Members should be admitted on new terms; and as the Society's premiums are calculated on the assumption of a reduction of 60 per cent. after seven years, the Directors recommend to the General Court to commence a Third Series, to be entitled to a reduction 15 per cent. less than the Second Series.

If this be done, it seems reasonable to expect that the *actual* reduction which new Members will have after *seven years*, will be about *two-thirds* of their premiums.

Anticipating that this recommendation will meet with the concurrence of the General Court, the Court of Directors beg further to report that all proposals for assurance made since the 30th June last have been received, subject to the regulations which the General Court shall adopt in respect of the reduction of premium.

In explanation of the above we give the following from the prospectus of the Association.

The Members* are entitled to a reduction of premium after seven years from the date of assurance.

The *rate* of reduction is determined every year by a valuation of the affairs of the Society, and being therefore liable to annual fluctuation, the amount of reduced premium payable may be more or less from year to year.

On policies effected before 1856 the rate of reduction has been as follows:—

60 per cent. in		1840.	70 per cent. in		1854.
62½	„	1841.	70	„	1855.
65	„	1842.	74	„	1856.
65	„	1843.	80	„	1857.
64	„	1844.	81	„	1858.
64	„	1845.	82	„	1859.
65	„	1846.	82½	„	1860.
66	„	1847.	83½	„	1861.
66	„	1848.	83½	„	1862.
67	„	1849.	84	„	1863.
68	„	1850.	84½	„	1864.
69	„	1851.	85	„	1865.
70	„	1852.	86	„	1866.
70	„	1853.	87	„	1867.

In 1825, 1830, and 1844, it was found necessary, for the protection of the interests of the older Members, to increase the rate of premium payable on new assurances; and, with the same object, in January, 1856, a *Second Series* was formed, to the Members of which the rate of reduction will always be 10 per cent. less than to those who had joined the Association before that date.

Again, on 1st July, 1865, a *Third Series* was commenced, to become entitled to a rate of reduction always 15 per cent. less than the rate to which the *Second Series* will be entitled.

A GENERAL STATEMENT *of the Affairs of the* **LONDON LIFE ASSOCIATION,** *estimated on* 30*th April,* 1865, *as up to the* 30*th June,* 1865.

	£	s.	d.
The present value of £4,210,313 assured on the lives of Members, 1st series	2,477,969	0	0
The present value of £2,614,490 assured on the lives of Members, 2nd series	1,042,521	0	0
The present value of £217,990 assured on lives not as Members .	107,473	0	0
Reserve for salaries, fees, and current expenses payable before 1st July, 1865	2,000	0	0
Claims ascertained but not yet paid	59,250	0	0
Reserve for claims which may have accrued, and have not been ascertained, and for claims which may accrue before the 1st July, 1865	35,994	0	0
£	3,725,207	0	0

* Members are those who assure their own lives at the rates of Premium in Table I.

	£	£	s.	d.
£70,000 3 per cent. Consols valued at	60,812			
120,000 New 3 per cents.	103,800			
10,000 Bank Stock	22,575			
111,500 Liverpool Corporation bonds	111,500			
Advanced on Mortgage and Policies	2,525,405			
House, furniture, fixtures, &c., in King William Street	14,919			
Policy and Mortgage stamps in hand	243			
Cash at the Bank of England	18,798			
Cash at the London and Westminster Bank	2,000			
		2,860,052	0	0
Various sums due to the Society before 1st July, 1865		69,564	0	0
The present value of £5,699, being the amount of annual premiums on assurances not as Members		57,188	0	0
The present value of such part of the annual premiums amounting to £95,606 on the lives of Members (2nd series), as they will be required to pay in full		225,481	0	0
The present value of the future reduced premiums on the lives of Members:—				
1st series at 85 per cent. reduction	222,375			
2nd series at 75 per cent. reduction	290,547			
		512,922	0	0
£		3,725,207	0	0

The Income of the Society is as follows:—

£190,000	3 per cent. Stock, producing yearly	£5,700
10,000	Bank Stock	1,125
111,500	Liverpool Corporation bonds	4,460
2,525,405	advanced on Mortgage and Policies	113,611
		£124,896

Gross amount of annual premiums on 5,642 existing Policies (30 June 1865) £249,228 5 10

CLERICAL, MEDICAL, AND GENERAL LIFE ASSURANCE SOCIETY.

Established 1824.

EIGHTH DIVISION OF PROFITS.

Report of the Directors.

In conformity with the provisions of the Deed of Constitution and of the Society's Special Act of Parliament, the Directors have called the present Meeting, in order to lay before the Proprietors and the Assured a statement of the transactions of the Society during the Quinquennial period which terminated on the 30th June 1866, and the result of an Investigation into its financial position at the same date.

The period under review embraces the years from the 38th to the 42nd inclusive of the Society's existence,—a portion of that period of full maturity during which the soundness of the basis on which an Assurance Society rests is put to the severest test. The results to be reported, derive, therefore, additional weight from this consideration.

To the successful progress of the Society during this period, the following facts bear ample testimony:

The new Assurances, which where purely English, were for a total sum of **£1,518,181**, producing Premiums amounting to **£50,497** annually, of which sums the former exceeds by £31,811, and the latter by £2,392, the corresponding items of the previous five years, although these constituted the largest amount of new business transacted in any like period.

The Income, notwithstanding a reduction in the Premiums in lieu of Bonus additions of over £5,600 per annum, rose from £195,400 on the 30th June 1861, to **£215,237** on the 30th June 1866, being an increase of nearly **£20,000** per annum.

The Assurance Fund, which in 1861 was £1,422,191, reached **£1,619,539** in 1866, showing an increase, after payment of £85,303 on account of Bonus at the last Division, of **£197,348**.

The Interest yielded during the whole period on all the Society's property, invested and uninvested, was on the average £4. 1*s*. 9*d*. per Cent., being somewhat in excess of that realized during the previous five years. The property invested yielded, on the 30th June 1866, **£4. 4*s*.** per Cent., a rate of Interest that will be deemed highly satisfactory when the unimpeachable character of the Securities is taken into account.

On the other hand, the mortality experienced during the five years has not been altogether so favourable as in the previous period. The Claims that accrued were under 856 Policies, amounting to £643,885, as against 700 Policies, for £464,280, in the previous five years, making the total Claims from the commencement £2,265,763. It is not, however, either in the number or the aggregate amount of these Claims that their unfavourable character is to be found, but in the circumstance that an unusual proportion of them fell upon Policies for large sums and of less than average duration. In stating this, the Directors believe themselves to be recording the common experience of the larger Assurance Institutions during the same period.

Passing now to the actual financial position of the Society on the 30th June, it may be stated that the Policies in force on that day were 8,331 in number, assuring, with their Bonus additions, the sum of £5,096,351; and these Policies have now to be dealt with.

It will be seen on reference to the annexed statement of Assets and Liabilities, that, after deducting the Proprietors' Capital of £50,000,

	£	*s.*	*d.*
The Assets on the 30th June last were .	1,619,539	14	8
And the Liabilities to the same date .	1,343,708	19	2
Leaving a Surplus of	£275,830	15	6

Deducting from this Surplus the sum of £50,000 required by the Society's Special Act of Parliament to be set aside as a permanent reserve fund, there remains available for Division the sum of £225,830. 15*s*. 6*d*., of which the Directors recommend the distribution of **£225,000**, being the nearest amount convenient for that purpose.

This sum, though less by £12,000 than that divided in 1862, exceeds by £30,000 the corresponding amount in 1857; and is fully 22 per Cent. of the total receipts from all sources during the five years—or considerably more than a whole year's Income.

It may not be uninteresting to add that the proportion of profit to income here indicated obtains almost exactly when the comparison is extended to the entire period of the Society's existence. To the 30th June last, the whole receipts from the commencement in 1824 were £4,722,730, whilst the actual profit to the same date was £1,012,830, or 21·5 per Cent. of such total Income.

Though the statement may be needless, the Directors think it well to record that in determining the liabilities the exact methods previously adopted have been rigorously adhered to. As on former occasions, the Carlisle Rate of Mortality, with 3 per cent. interest, has been employed, and none but the net premiums having been taken into account, every encroachment on, or anticipation of, future profits has been scrupulously avoided. The sum to be divided is, therefore, emphatically *realized profit*, which, fairly earned, is rightly divided.

Of the sum to be apportioned, it is known that one-sixth, or £37,500, falls to the Proprietors. The remaining five-sixths, or **£187,500**, fall to the assured, yielding a Reversionary Addition to the Policies of **£272,682**, in which sum every Policy on the participating scale of Premium, existing on the 30th June last, will share in proportion to its contributions to the funds of the Society since the last division.

This Reversionary Bonus will average nearly **45** per Cent., or vary, with the different ages, from **32** to **85** per Cent. on the Premiums received in the Quinquennial period on all the Policies among which it will be distributed, whilst its equivalent value in Cash will average over **26** per Cent. of the like payments.

The high per centage again exhibited by the Cash Bonus induces the Directors to draw more marked attention than they have hitherto done to its real character and benefit. Of every £100 paid as premiums by a participating Policyholder, £20 is a marginal addition to the net or mathematical premium of £80 required for the actual risk, and is added, partly for expenses and undetermined fluctuations, and partly for the privilege of sharing in the general profits of the office. A return in cash, such as is now offered, averaging £26 for every £100 so paid as premiums, not only refunds to the assured the whole of such marginal addition, but leaves to be received by him a further sum of £6 as his share of profits that have virtually cost him nothing.

The last statement the Directors think it necessary to make has reference to the number of Policyholders who, retaining all their present right of sharing in future profits, may now relieve themselves of any further payment on account of premiums, by the surrender in whole or in part of the Bonus additions to their policies. This number is **350**; in 1862 it was 176. It would have been larger on this occasion by 194, had not some portion of the Bonuses on these Policies been taken in Cash or been otherwise appropriated.

ASSETS and LIABILITIES on 30th June, 1866.

ASSETS.

	£	s.	d.
Funded property, viz., £255,000 Stock	234,170	11	0
Mortgages	1,185,097	12	4
Advances on Life Interests	134,398	0	0
Freehold House for the Society's Offices	12,750	0	0
Value of Bonuses on Policies belonging to the Society at other Offices	14,130	11	4
Premiums, Dividends on Stock, and proportion of Interest due	45,162	6	3
Agents' Quarterly Balances	15,668	8	6
Balance at London and Westminster Bank:—			
Deposit Account	15,000	0	0
Drawing Account	12,937	7	2
Cash in the Office	224	18	1
Total Assets	1,669,539	14	8
Deduct Proprietors' Guarantee Fund	50,000	0	0
Consolidated or Assurance Fund	1,619,539	14	8

LIABILITIES.

	£	s.	d.	£	s.	d.
Value of Policies effected on the Participating Scale	998,102	7	0			
Value of existing Bonuses	225,353	5	0			
Value of Policies effected on the Non-participating Scale	61,738	6	4			
Value of Annuities	733	17	0			
Dividends due	2,562	10	0			
Claims by Deaths which occurred before 30th June, 1866, unpaid	53,199	18	0			
Due for Rates, Income Tax, Commission, and sundry Expenses	2,018	15	10			
				1,343,708	19	2
Surplus				275,830	15	6
Deduct Reserve Fund, pursuant to section 32 of the Society's special Act of Parliament				50,000	0	0
Available for Division				**£225,830**	**15**	**6**

LEGAL AND GENERAL LIFE ASSURANCE SOCIETY.

Established 1836.

REPORT OF THE DIRECTORS.

THIS MEETING, called in pursuance of the provisions of the Deed of Settlement for the purpose of declaring the amount of Profit to be set apart out of the Assurance Fund in respect of the period of Five years ending with the 31st December, 1866, is the first held under the new system of distribution of the Profits of the Society.

The following is the result of a careful valuation of the Assurance Fund, made as a basis for such Declaration; viz.—

	£.	s.	d.
Value of Assets at 31st December 1866	1,382,843	–	–
Value of Liabilities at same date . .	1,203,990	–	–
Excess of Assets	178,853	–	–
Deduct for Reserve	8,853	–	–
Divisible Surplus	£170,000	–	–

The Directors recommend that this sum of £170,000 should be divided between the Proprietors and the Assured in the proportions fixed by the present provisions of the Deed of Settlement; viz.—

To the Proprietors, **one Tenth** part, being £17,000.
And to the Assured, **nine Tenth** parts, being £153,000.

After such apportionment has been made, the respective amounts of the two Funds as on the 1st January 1867, will be—

	£.	s.	d.
Proprietors' Fund	177,710	5	1
Assurance Fund	1,219,422	1	–

The Constitution of the Society requires that the Proprietors' Fund shall never, by the declaration of a Dividend, be diminished below £160,935. 19.

	£.	s.	d.	
Therefore, of the sum of	177,710	5	1	above mentioned,
reserving such . . .	160,935	19	–	
there remains . . .	£16,774	6	1	available towards

an increase of the Proprietors' Dividend during the current period of Five years.

This will allow an addition of 3*s.* 6*d.* to be made to the Dividend last declared of 6*s.* per Share per annum.

It is therefore proposed by the Directors to declare a Dividend of 9*s.* 6*d.* per Share, payable in July annually until the next declaration of Profits. The Shares remain as before, with £8 paid-up on each.

The Share of Profit allotted to the Assured (£153,000) exceeds by £17,000 the amount they would receive, if the former system of distribution were now in force. It will allow a substantial increase in the rate of Bonus on all Participating Policies over that last declared, whether taken as a Reversionary Addition to the sum assured, or as a reduction in the Premium.

This Share will be converted, as usual, into a Reversionary Bonus payable at death, and attaching at their renewal date in 1867 to all Bonus Policies which shall then have endured for Five full years, and at the date on which they complete such Five years in all other cases.

A Prospective Annual Bonus at the same rate, and equally provided out of the Profits now realised, will, in addition, be assigned to all Bonus Policies becoming Claims by death after not less than Five years duration before the next division of Profits takes effect. This regulation secures to

every Policy entitled to participate in the Profits a full Bonus for each complete year it may have been in force when the Life Assured fails.

The date at which this Declaration is made renders it impossible to announce at present, in detail, the exact addition to be made to each individual Policy. This will be done, by letter, as early as practicable. The following Table will, however, enable each Policy-Holder to anticipate very closely the result in his own case. It should be borne in mind, in using the Table, that a Bonus at the same rate is given upon the amount of all previous Bonus remaining attached to the Policy.

SPECIMENS of REVERSIONARY BONUS added to Policies at the Fifth Division of Profits (1866) in respect of each £1,000 assured, an equal Rate of Addition being given upon all previous Bonus continuing attached to Policies:—

Age at Admission.	Annual Premium for £1,000 Assured.	Vested Reversionary Bonus in respect of Five Years complete.	Prospective Annual Bonus until next Division of Profits.
	£. s. d.	£. s. d.	£. s. d.
20	20 3 4	52 10 –	10 10 –
25	22 10 10	55 – –	11 – –
30	25 7 6	57 10 –	11 10 –
35	28 15 10	60 – –	12 – –
40	32 19 2	67 10 –	13 10 –
45	38 5 10	72 10 –	14 10 –
50	45 7 6	72 10 –	14 10 –

At the option of the Assured the Reversionary Bonus may, when vested, either (1st) be surrendered for its then present value in cash, or, (2ndly) together with the Annual Prospective Bonus, be applied in reduction of the full Annual Premium for the term of years until the next Division of Profits. At the last Division such reduction, where available for the full term of Five years, was at the rate of 25 per cent. upon the Premium. The Directors are happy in announcing that the present Bonus will allow 27½ per cent. in similar cases.

It remains to state the principles upon which the valuation has been conducted. First, then, the duration of the Lives assured has been estimated by the Table of Mortality in use by the Office for thirty years past, and which indicates at every age a mortality One-fifth greater than that indicated by the well-known and widely-adopted "Carlisle Table." Secondly, The rate of Interest for Money has been taken at 3 per cent. per annum. Thirdly, The Net Premiums alone have been brought into account, to the exclusion of the Charges for Agency and Expenses contained in the Gross Premiums actually receivable.

The result of thus valuing separately each of 3,082 Policies, assuring an aggregate amount (inclusive of Bonus) of £3,911,728, shows a present liability in respect of these Policies of £1,087,920.

The Assets available against this liability have been estimated at their net market value on the 31st December 1866.

CROWN LIFE ASSURANCE COMPANY.

Established 1825.

QUINQUENNIAL REPORT, 1865.

AT an Extraordinary General Court, held on the 27th August 1858, it was resolved that the Profits of the Company should be divided quinquenially, instead of septennially, from and after the year 1860, when the fifth septennial division of the Profits was made.

In accordance with that Resolution, the Profits realized during the five years which ended on the 25th March 1865, have now to be divided, and the Directors have much pleasure in reporting to the Proprietors and the Assured the result of the investigation of the Company's affairs as they stood on the 25th March last.

As to the Progress of the Business.

During the last five years the Company has issued **2,359 Policies,** assuring the sum of **£1,225,643.** A comparison of these figures, with the new business of the immediately preceding septennial period, shows an *Annual increase*, in the last five years, of 124 in the number of the Policies, and of £61,368 in the sum assured.

As to the Mortality—Actual and Expected.

The claims during the five years have accrued under 527 Policies, assuring the capital sum of £319,616. Of these claims, the Policies that were entitled to share in the Profits assured the capital sum of £252,134, and the Bonuses paid thereon amounted to £50,820. The *average* Bonus paid in the period was therefore rather more than 20 per cent. on the sum assured.

The *expected* claims, during the five years, have been carefully calculated according to the "Equitable" Experience Mortality Table, from which the Company's premiums are deducted; and the result very nearly agrees with the actual mortality in the period. The *expected* number of claims was 523; the *actual* number 527. The *expected* amount of the claims was £331,599; the *actual* amount £319,616. The actual mortality was, therefore, at the rate of nearly 1,008 deaths for each 1,000 expected; and the actual amount of the claims, at the rate of £964 for each £1,000 expected.

The amount of claims in the period has not been larger than was expected, but more than the usual proportion of deaths has fallen on Policies of recent date for large amounts; and, consequently, the profits of the period have been reduced below the rate that would have been realized if the ordinary average had obtained. Such casualties are to be expected in the experience of every office; and the change to *quinquennial* divisions, while it secures to the Assured more frequent additions to their Policies, will probably cause greater fluctuations in the rate of profit to be periodically divided, than existed when the Bonuses were allotted septennially.

As to the Investments.

The average rate of interest realized on the whole of the Investments was, on the 25th March last, £4 : 9 : 6 per cent. per annum.

As to the Position of the Company.

The total **Policies in force**, on the 25th March 1865, were **5,338**, assuring the capital sum of **£3,006,242**.

The total **Annual Income**, on the 25th March last, was, from Premiums £94,389; from Interest £45,107; together **£139,496**.

The total **Net Funds**, on the 25th March last, comprising the Assurance Fund and the Proprietors' Guarantee Fund, amounted to **£1,014,125**.

As to the Principles of the Valuation.

The Table of Mortality, employed in all the Valuations on this occasion, is the same as was used at the five preceding septennial investigations, namely—the "Equitable" Experience Table.

The *net* rate of interest realized from the investment of the Assurance Fund, during the last five years, has averaged about $4\frac{1}{4}$ per cent., but the rate of interest assumed in the valuation of the Policies of Assurance is only $3\frac{1}{2}$ per cent.; so that whatever interest may hereafter be realized beyond $3\frac{1}{2}$ per cent. will go to increase the Profits to be apportioned at future divisions.

The rate of interest assumed in valuing the Annuity transactions and the Reversionary Bonuses on Policies is 4 per cent.

In estimating the liability under the Policies, the whole of the *Loading*, amounting on the existing Assurances to £21,240 *per annum*, has been thrown off; and credit has therefore been taken in the account for only the present value of the future *pure* or net premiums, so that none of the profits to be hereafter realized have been anticipated.

It is upon these data that each Policy has been valued separately; and the results have been carefully checked.

STATEMENT OF THE AFFAIRS OF THE COMPANY.

Showing the Amount of Assets and Liabilities, and the Amount of the Surplus Fund for Division, as at the 25th March, 1865.

ASSETS.

	£	s.	d.	£	s.	d.
I. FUNDS IN EXPECTATION—						
Present value of *Re-assurances* for £137,300				14,638	9	3
Interest on Claims not due, from 25th March, to dates when payable				140	11	1
				£14,779	0	4
II. FUNDS IN POSSESSION—						
Government Securites	£64,688	5	5			
Mortgages	373,416	4	9			
Railway Debentures	250,912	10	0			
London Water Companies' Debentures	30,385	7	6			
Dock Companies' Debentures	50,000	0	0			
Loans on the Company's Policies (within their Surrender Value)	38,622	9	1			
London and Agency Premises	9,256	11	11			
Interest due and accrued to the 25th March	10,069	12	11			
Premiums, due at Head Office and in Agencies, the time for payment of which has not expired	22,854	14	3			
Bills receivable and Policy Stamps in hand	933	15	8			
Cash at Bankers, Current and Deposit Accounts	6,760	12	5			
				857,900	3	11
				£872,679	4	3

LIABILITIES.

		£	s.	d.
I. In Expectation—				
Present Value of the Company's Policies, assuring the capital sum of £3,006,242 : 12 : 0		620,885	18	0
Present Value of £208,903 Bonuses declared at former Divisions, now remaining		123,988	12	0
Present Value of £819 : 10 : 2, future annual reductions of Premiums for Life		7,077	11	1
Present Value of £2,161 : 4 : 9 Annuities now payable . .		16,871	4	0
Present Value of £1,136 Deferred and Contingent Survivorship Annuities		1,028	6	0
		769,851	11	1
II. Immediate—				
Amount of Claims admitted, but not due till after 25th March	£22,204 10 0			
Premiums and Interest paid, but not due till after 25th March	532 7 9			
Annuities due, but not paid till after 25th March	229 12 10			
		22,966	10	7
Total value of the Liabilities		£792,818	1	8
Balance, being the amount of the Surplus Fund for Division, realized in the Five Years which ended on 25th March 1865 . . .		79,861	2	7
		£872,679	4	3

As to the Distribution of the Surplus Fund.

		£	s.	d.
The total Cash Surplus for Division is shown by the preceding statement to be		£79,861	2	7
Of which, in accordance with the provisions in the Deed of Settlement, there have to be applied in augmentation of the Proprietors' Guarantee Fund, which constitutes a permanent security to the Assured,				
One-third of the surplus applicable to Policies issued *before* 25th March 1860 .	£21,887 12 0			
One-sixth of the surplus applicable to Policies issued *after* 25th March 1860	£2,366 8 0			
Together . .		£24,254	0	0
Leaving the Cash Surplus to be apportioned among the Assured whose Policies were in existence on the 25th March 1865		£55,607	2	7
And which sum will be apportioned in the following manner, namely:—				
Among the Assured, under 3,040 Policies for £1,808,479, dated *before* the 25th March 1860.		£43,775	2	7
Among the Assured, under 1,738 Policies for £802,220, dated *after* the 25th March 1860 .		£11,832	0	0
Sum as before		£55,607	2	7

The sum of £24,254 now to be added to the Guarantee Fund, and the sum of £1,117, being the Proprietors' share of the Intermediate Bonus accrued in the last five years, making together the sum of £25,371, will enable the Directors to declare a Bonus of £4 : 2 : 0 on each Share, retaining a balance of £115 unappropriated. The fund, which in 1860 consisted of 6,160 Shares at £26 : 10 : 0, and amounted to £163,240, will, after the apportionment, consist of 6,160 Shares at £30 : 12 : 0, and amount to £188,496.

The sum of £55,607 now to be apportioned among the Assured, represents Reversionary Bonuses amounting to £111,076, and the sums assured by the Participating Policies in force on the 25th March last, have accordingly been increased to that extent.

The Bonuses allotted to the individual Policies vary in amount according to the respective ages of the Assured and the duration of their Policies, and the usual Certificate, informing each Policy-holder of the amount of the Bonus that has been added to his Policy, will be prepared and forwarded as soon as possible.

Regulations as to the Application of the Bonus.

The Bonuses allotted to Policies of more than two years' standing on the 25th March 1865, are *immediately vested;* and in the case of Policies on which 3 years' premiums were not paid at that date, the Bonuses will become vested when the third year's premium shall have been paid.

In the event of any Policies, on which not less than three years' premiums were paid before the date of death, having become claims since the 25th March 1865, the Bonus now declared on such Policies will be paid to the representatives of the Assured.

The Assured have the option of applying the Bonuses to augment the sum in their Policies; or they may receive the value of their vested Bonuses in cash, or apply it towards the reduction of their annual premiums, for life, or for a fixed number of years until the next division of profits, which will be in the year 1870. A Table, showing the values of the Bonuses, at the various ages, if dealt with in any of these ways, will be printed on the Bonus Certificates which will be sent to the Policy-holders.

As to Policies, dated before 25th March 1860, which receive Two-thirds of the Profits.

The practice of the Company for the first thirty-five years of its existence, that is from the year 1825 to 1860, was to divide among the Assured *Two-thirds* only of the ascertained surplus at the end of every *Seventh* year. The Participating Policies that were issued before the 25th March 1860, therefore, received at each Septennial Division, during their existence, two-thirds of the entire profits arising from all classes of the business. And, accordingly, those Policies have received at this the first Quinquennial Division (and will receive at the future divisions which they may survive) two-thirds of the entire profits realized from all classes of the Policies dated before the 25th March 1860. With a view to afford to the Assured the means of comparing the results of the two last divisions, a careful calculation has been made by the Actuary, by which it has been ascertained that the sum required to provide a Bonus now, at the same rate per cent. per annum as was declared at the last Septennial Division in

1860, is £50,560; whereas the two-thirds allotted at this Quinquennial Division, as has been already stated, amount to only £43,775. This apparent falling off in the amount of the present divisible fund, has not, however, been caused by the Company having made less profit in the quinquennial period than formerly; but is owing to the new regulations that have come into operation since the year 1860, and have conferred advantages on the whole body of the Assured (1) by giving to all who die in the interval between the periodical divisions, a share of the accruing profits proportioned to the number of premiums paid since the immediately preceding division; and (2) by giving Bonuses to the larger number of the Assured who survive the *fifth* year, instead of, as formerly, postponing the division till the end of the *seventh* year, and then dividing the larger accumulated profit among the smaller number of survivors. Those changes have affected the fund at this Quinquennial Division in the following manner:—

1. The Intermediate Bonuses that have been paid on the Policies which have become claims in the last five years (such Bonuses not having been paid in former periods) amount to	£2,616	0	0
2. The Quinquennial Fund of £43,775 having been divided *two years earlier* than at former divisions, the Assured thus gain the interest on the Fund for these two years, which, reckoned at only 4 per cent. (though the Company would have realised £4 : 7 : 0 per cent.,) amounts to	3,572	0	0
3. The Policies which, if the Division were postponed till the end of the seventh year, as formerly, would lapse by death in the next two years, and thus *have no claim for Bonus on the Septennial Fund*, amount to £106,000, on which sum, however, the *five* years' cash profit has now been allotted, and amounts to	2,773	0	0
These three sums amount to	£8,961	0	0

and in order that a correct comparison may be made of the results of the two last periods, that amount should be added to the Quinquennial Fund.

We have, therefore, the Fund actually divided at the end of the fifth year.	£43,775	0	0
And the sums paid out of, or deducted from, the five years' profits as already shown	8,961	0	0
Making the total sum, or value, received at this division in 1865, by the Assured whose Policies were in force in 1860	£52,736	0	0

So that the Profits of the Company with respect to those Policies have been £2,176 *more* in the five years to 1865, than the sum divided, on an equal amount Assured by Policies of like duration, at the last Division in 1860.

As to Policies, dated after 25th March 1860, which receive Five-sixths of the Profits.

It was resolved that all Assurers who effected Policies in the Participating Class, after the 25th March 1860, should receive *five-sixths* of the entire profits to arise from all classes of the Assurances completed after that date. Of the Assured who have joined the Company since the 25th March, 1860, there are 1,738 whose Policies are in the Participation Class, and were in force on the 25th March 1865, and who accordingly shared in the Profits at this Division for the first time. The Bonuses which have been allotted to their Policies are stated in the following Table; and, for comparison, the Bonuses that were declared on Policies of like duration and amount at the last Division in 1860 are also stated:—

TABLE OF BONUS ADDITIONS

To Policies effected since the date of the last Division of Profits on the 25th March 1860.

Policies effected before 25th March	Number of Premiums paid.	Sum Assured.	**Bonus, 1865.**	Bonus, 1860
1861	5	£1000	£68	£60
1862	4	£1000	£54	£48
1863	3	£1000	£41	£36
1864	2	£1000	£27	£24
1865	1	£1000	£14	£12

The Bonuses allotted at this Division, to Policies which have been effected in the last five years, are therefore 13 per cent. *larger* than the Bonuses allotted, at the last Division in 1860, to Policies of equal duration and amount.

PRACTICAL QUESTIONS.

SUGGESTIONS have been made at various times that the usefulness and interest of this *Journal* might be much increased if greater prominence were given to questions of a practical character; and it has been suggested in particular, that solutions might with advantage be inserted of some of the more difficult cases that come before actuaries in the course of their practice. We are therefore induced to give the particulars of the two following cases, with the approximate solutions which appeared sufficiently accurate to the actuary to whom the questions were submitted. We shall be glad if such of our readers as may have leisure will give more exact solutions; and we shall also be glad to receive other approximate solutions from persons who may consider the following unsatisfactory.

Question 1. "The sums of £80,000 Consols and £8,000 New Three per Cent Stock, are settled on the following trusts:—

"An annuity of £500 is to be paid to a gentleman, A, aged 75, for his life; and on his death a sum of £1000 is to be paid to each of his five children. Three annuities of £250 each are to be paid to three ladies, B, C, D, aged 80, 76, and 71 for their respective lives; and on the death of B, £1000 is to be paid to each of her three children; and on the death of D, £1000 to each of her four children. Subject to these payments, the surplus income of the Trust Fund is to be invested in similar stock as received, and to accumulate until the death of the last survivor of the four annuitants, when the capital is to be divided.

"X is absolutely entitled on the death of such last survivor to one 20th part of the accumulated sum, and to one 7th of another 20th part; and wishes to know what sum he can borrow on security of his share."

The present annual income of the fund being £2640, ($=3 \times 880$) and the annuities amounting to £1250, there is a present annual surplus of £1390; or rather less, when income tax is taken off; and the total amount of the legacies to be paid is £12,000.

Consider first what amount of stock will suffice to pay these legacies. Taking stock at the mean minimum price of 85, it would require £14,117·65 Stock to pay the legacies of £12,000; and the annual income of this stock (£420·529) will be available towards payment of the annuities to A, B, and D.

We first then set apart the above amount of stock—£14,117·65,—for payment of the legacies, and part payment of the annuities; and we then seek what further amount of stock must be set apart to provide for payment of the balance of the annuities. We must consider separately the cases of A B C and D. On the death of A (age 75) there is payable £5000, or say £5882·36 Stock, the annual income from which is £176·471. The balance of the annuity is £323·529 (=£500−£176·471). On the death of B (age 80) £3000 is payable—say £3529·41 Stock—the income of which is £105·882; and the balance of B's annuity is therefore £144·118. The annuity for the life of C (age 76) is £250. On the death of D (age 71) £4000 is payable, or say £4705·88 Stock—the income of which is £151·176; and the balance of D's annuity is £108·824.

We have therefore to calculate the amount of stock that will provide for the following annuities.

£323·529	on life of	75	the value of which according to the Carlisle $3\frac{1}{2}$ per cent tables is found to be	1899·9
144·118	"	80	" "	687·7
250·000	"	76	" "	1411·9
108·824	"	71	" "	766·4
826·471				4765·9

Since we take stock at 85, we may assume that £100 sterling produces interest at the rate of $3·529 \left(=\frac{3}{·85}\right)$ per cent per annum; or we may value the above annuities at $3\frac{1}{2}$ per cent interest. Taking then the values of $a_{75}+·5$, $a_{80}+·5$ &c. by the Carlisle $3\frac{1}{2}$ per cent tables, we get the values of the annuities as set out above—the total value being £4765·9

sterling, or, say, £5606·9 Stock. We thus conclude that this amount of stock will provide for the balance of the annuities. Adding to this, the amount obtained above, £14,117·6, we conclude that a sum of £19,724·5 Stock will provide for payment of all the annuities and legacies. This leaves a sum of £68,275·5 Stock, which is to accumulate at compound interest until the death of the last survivor of A, B, C, D.

We have now to investigate a formula applicable to this case. Suppose that £1 is invested at compound interest at the rate j until the termination of a given status, so that if the status terminate in the 1st, 2nd, 3rd, . . . , nth, years respectively, there will be receivable the sums, $1+j$, $(1+j)^2$, $(1+j)^3$, ... $(1+j)^n$; and let the value of the reversion be estimated at the rate of interest i; then if $q_1, q_2, \ldots q_n$ are the probabilities of the status terminating in the 1st, 2nd, . . . nth, years, the value of the reversion will be

$$q_1\frac{1+j}{1+i}+q_2\frac{(1+j)^2}{(1+i)^2}+\ \ldots\ +q_n\frac{(1+j)^n}{(1+i)^n}+\ \ldots .$$

But this is the value of £1 to be received at the termination of the given status, estimated at a rate of interest I, such that

$$\frac{1}{1+\mathrm{I}}=\frac{1+j}{1+i};\ \text{or}\ \mathrm{I}=\frac{i-j}{1+j}.$$

Now to apply this formula to the case under consideration, we must first determine at what rate of interest the stock may be considered to accumulate. Each £100 stock produces annually £3 *sterling* interest, which is to be invested, say at the price 95, and will therefore produce $\frac{3}{\cdot 95}=3\cdot 158$ *Stock*. If then we make $j=\cdot 03158$, and $\mathrm{I}=\cdot 03$ (because this is the rate at which we have tables calculated) we shall have

$$i=(1+\mathrm{I})(1+j)-1=\cdot 0625$$

and we shall estimate the value of the reversion at $6\frac{1}{4}$ per cent—a very fair rate. We have thus next to find the value (by the Carlisle 3 per cent tables) of the reversion to £1 on the death of the last survivor of A, B, C, D; or 80, 76, 75, 71. Thus

$$\begin{array}{rlrl}
a_{80}= & 4\cdot 365 & a_{75}= & 5\cdot 512 \\
a_{76}= & 5\cdot 277 & a_{71}= & 6\cdot 737 \\
\hline
 & 9\cdot 642 & & 12\cdot 249 \\
a_{80.76}= & 2\cdot 728 & a_{75.71}= & 3\cdot 672 \\
\hline
a_{\overline{80\cdot 76}}= & 6\cdot 914=a_{71} & a_{\overline{75\cdot 71}}= & 8\cdot 577=a_{66} \\
a_{\overline{75\cdot 71}}= & 8\cdot 577=a_{66} & & \\
\hline
 & 15\cdot 491 & & \\
a_{71.66}= & 4\cdot 882 & & \\
\hline
a_{\overline{80\cdot 76\cdot 75\cdot 71}}= & 10\cdot 609 & & \\
\mathrm{A}_{\overline{80\cdot 76\cdot 75\cdot 71}}= & \cdot 66188 & &
\end{array}$$

Hence the value of the reversion to £68,275·5 Stock expectant on the death of the last survivor is £45,190 stock; or £38,411·5 sterling, if we again take stock at 85, since it is now a question as to the selling price. Lastly, the share of X, ($\frac{1}{20}$th + $\frac{1}{7}$ of $\frac{1}{20}$th) will be of the value £2195.

It must be borne in mind that the share of X is probably subject to legacy or succession duty; and that a double duty is probably payable on the $\frac{1}{7}$th of the $\frac{1}{20}$th share. Nor must it be lost sight of that accumulation for more than 21 years after the death of a testator is illegal. Altogether, we may conclude that unless the legacy duty is very heavy, the reversion might be expected to sell for about £2000.

INSTITUTE OF ACTUARIES.

SOLUTIONS OF THE SECOND YEAR'S EXAMINATION QUESTIONS.

(Continued from vol. xiii., p. 260.)

Questions for 1864.

1. Find an expression for the Napierian logarithm of any number in terms of the logarithm of the preceding number; and by the aid of such expression find $\log_{10}2$ to five decimal places.

2. To what base is $-1·765776$ the log. of 1,000? Given reciprocal of $1·765776 = ·566323$.

Ans. Base $= \dfrac{1}{50}$.

3. Find the probability of turning up 1, 2, 5, in any order, in three successive throws of a die. Explain the principles assumed in the solution of the question.

Ans. $\dfrac{1}{36}$.

4. A and B play together as follows:—A throws a coin and continues to throw as long as he throws head. If this happen four times, he wins; if not, he resigns the coin to B on the same terms, B to resign it again if he should fail, and so on. The first who throws four heads in succession wins one shilling. Find the values of the expectations of A and B.

Ans. $\dfrac{64}{127}$ and $\dfrac{63}{127}$.

5. Find the relation between the discount of any sum payable n years hence, and the present value of an annuity for the same term of years. Distinguish between the discount here referred to and commercial discount, as well as discount at simple interest.

Ans. If D be the discount, A the annuity, $A = \dfrac{D}{i}$.

6. Construct a specimen table of the present value of an annuity, at 3 per cent. interest, of £1 for 25, 26, &c., years, to 30 years. Given $v^{30} = ·4120$.

7. Give some account of the Carlisle table of mortality. What is to be remarked as to the relative proportion of the sexes among the lives on

which it was based? How does that affect the use of the table for the purposes of a Life Office?

Ans. The Carlisle Table of Mortality was constructed by Milne from observations made by Dr. Heysham, giving, in intervals of age, the population of two parishes in that city in January 1780 and December 1787, and also the deaths in the same parishes in the nine years 1779–87.

The population enumerated in 1780 was 7677, whereof 3491 were males and 4186 females. The relative proportions, 45 per cent of males, and 55 per cent of females, show an unusual preponderance of the latter sex. Now we know from other observations that the mortality is more favourable at the older ages among females than males. If therefore, as is sometimes contended, the Carlisle Table fairly represents the mortality likely to prevail in any English community, it follows that in Life Offices, where the proportion of males to females is about 10 to 1, this table will understate the mortality likely to be experienced at the advanced periods of life.

8. What are the two different modes by which the law of mortality of a population has been ascertained?

9. The function p_x being tabulated, show how to construct from it a table of the mean duration of life.

Ans. $e_x=\frac{1}{2}+p_x+p_x p_{x+1}+p_x p_{x+1} p_{x+2}+\ldots$ or $e_x=\frac{1}{2}+p_x(\frac{1}{2}+e_{x+1})$.

10. Let P represent the population of a country at a given period, $\frac{1}{a}$ the proportion of deaths in a year, and $\frac{1}{b}$ the proportion of births in a year; find the population at the end of n years, assuming that no emigration or immigration takes place.

Ex.—Given the death rate 1 in 25, birth rate 1 in 20, in about how many years will a population double itself?

Ans. (1) $P\left(1-\frac{1}{a}+\frac{1}{b}\right)^n$. (2) About 70.

11. Describe briefly commutation tables, and explain the mode of constructing Column D for joint lives.

12. Prove the formula $M_x=vN_{x-1}-N_x$.

13. Distinguish between "probable lifetime" and "mean after-lifetime," as defined by Dr. Farr; and state upon what hypothesis they are identical.

14. Why is not the value of an annuity on the life of a person of a given age equal to the value of an annuity certain for the term of years representing the mean duration of life at that age?

Vide De Morgan's "Budget of Paradoxes," *Journal of the Institute*, vol. xii., pp. 32, 33; Jones on *Annuities*, vol. i., pp. 122, 123.

15. If B = present value of a benefit of £1 upon a given life (x),
B_1 = do. do. do. ($x+1$),
p = the probability of a payment of B being received in the first year, and
Π = the probability of x surviving the year,

Prove that $B=v\Pi\left(\frac{p}{\Pi}+B_1\right)$.

16. Show how to apply the equation in the last question to the construction of a table of the value of a reversion of £1 on decease of a single life.

Ans. $A_x = vp_x\left\{\frac{1-p_x}{p_x} + A_{x+1}\right\}$.

17. An annuity on the longest of three lives, A, B, and C, is to be enjoyed by A for his life; after his decease it is to be divided equally between B and C during their joint lives, and the survivor of them is to have the whole. Find an expression for the value of B's interest.

18. Prove that the single and annual premiums for an endowment assurance are respectively $1-(1-v)(1+a_{x\overline{t-1}|})$ and $\frac{1}{1+a_{x\overline{t-1}|}}-(1-v)$.

19. Obtain an expression for the value of a life annuity in terms of the annual premium. Explain the modification of the formula when the premium is increased by a loading.

Ans. $\varpi_x = \frac{1}{1+a_x} - d \quad \therefore \quad a_x = \frac{1}{\varpi_x + d} - 1.$

If the premium be increased by a loading, ϖ_x will represent the Office premium, and a_x the value of an annuity on an isolated life aged x, such as to yield a purchaser interest at a rate corresponding to d, and enable him also to secure the return of his capital by a Life Assurance.

20. Find the single premium for an assurance for n years of £1 on the failure of a life aged (x), provided that another aged (y) survive him.

Ans. $_{\overline{n}|}A^{1}_{xy} = \frac{1}{2}\left\{_{\overline{n}|}A_{xy} + \frac{_{\overline{n}|}a_{x-1.y}}{p_{x-1}} - \frac{_{\overline{n}|}a_{x.y-1}}{p_{y-1}}\right\}$.

21. Deduce an expression for the annual premium for a deferred annuity, with return of all the premiums in the event of death occurring before the annuity is entered upon.

22. If a_x be the value of an annuity of £1 payable annually on a life aged (x), prove that the value of the annuity payable half-yearly is $a_x + \frac{1}{4}$ very nearly.

23. An assurance for the whole of life, at the ordinary premium, has been in force for n years. It is proposed to commute the future premiums for an annual premium payable for m years only. How is this to be calculated?

Ans. The premium required $= \frac{N_{x+n-1}}{N_{x+n-1} - N_{x+n+m-1}} \times P_x$.

24. A copyhold estate is held on three lives aged respectively 61, 50, and 45, each renewable at the end of the year in which it drops by a life of 7 years of age on payment of a fine of £5. Give an expression for the present value of all the fines for ever.

Ans. $(A_{61} + A_{50} + A_{45})\frac{5}{1-A_7}$.

25. Under certain Acts of Parliament for the improvement of land, power is given to borrow money in consideration of an annual rent-charge for a term of years certain. In such a case, point out a method of finding, at the end of any year, the amount of capital outstanding; and also of apportioning the rent-charge in any year between the interest and capital accounts.

Ans. If n be the term of years, then at the end of m years the capital outstanding is $\frac{1}{i}\{1-v^{n-m}\}$, the annual rent-charge being 1. Also of the rent-charge received at the end of the mth year, v^{n-m+1} is capital and $1-v^{n-m+1}$ is interest.

Questions for 1865.

1. Explain what is meant by the common and by the Napierian logarithm of any number.

Write down the series giving their values, and deduce a series adapted for actual computation.

2. Explain how a complete table of common logarithms may be computed to any number of decimal places required.

3. Approximate to the value of $\log_{10}2$ by means of a continued fraction.

4. Show that the logarithms of all numbers may be found to any required number of decimal places by means of a table giving to the required number of places the logarithms of

$$N,\ 1+\cdot 1\times N,\ 1+\cdot 01\times N,\ 1+\cdot 001\times N,\ \ldots .$$

for values of N from 1 to 9. If n places of decimals are required, what will be the last of the above quantities which it will be necessary to tabulate?

5. If two events are independent of each other, and the chance of the first happening is p_1, and the chance of the second p_2; prove that the chance of their both happening is p_1p_2.

Find the chance of both failing; also the chance of one happening and the other failing.

Ans. (1) $(1-p_1)(1-p_2)$; (2) $p_1+p_2-2p_1p_2$.

6. Find the value of a sum of money to be received at the end of n years if a certain event happen of which the probability is p.

Ans. $p(1+i)^{-n}$, or pv^n.

7. Explain De Moivre's hypothesis.

Assuming the truth of that hypothesis, find the expectation of life at any age: prove that it is also the term which there is an even chance of surviving; and find at what age it is most probable a person of a given age will die.

Ans. (1) $l_x=86-x$; (2) $e_x=\frac{86-x}{2}$; (3) all ages are equally probable.

8. With which table of mortality does De Moivre's hypothesis best agree? In what respects do the tables generally in use differ from the hypothesis?

Ans. (1) The Northampton. (2) In the tables in general use, the decrements, instead of being constant, increase (after childhood) up to about the age 72, and then decrease.

9. If a mortality table were such that the numbers living formed a series in geometrical progression, what would be the nature of the tables giving the chance of dying in a year, and the expectation at any age?

Ans. The chance of dying in a year would be constant, or the same at all ages. (2) If $l_{x+1}=kl_x$, and $x+z$ be the extreme age in the table, then $e_x=\frac{1+k}{2(1-k)}-\frac{k^{z+1}}{1-k}$.

10. Describe Gompertz's theoretical law of mortality; and give his formula for the number living at any age.

Ans. $l_x=gd^{p^x}$.

How is this theory found to agree with observation?

Ans. Very closely, for a considerable number of years; but the same values of the constants will not apply throughout the table.

11. Prove the formula for an annuity on the last of three lives.

12. Explain how such an annuity is found in practice; and show that the rule is accurately true if the mortality agrees with Gompertz's hypothesis.

13. Find the single premium for an assurance on the life of x, provided he die during the life of y or within t years after the death of y.

Ans. $A_x-\frac{D_{x+t}}{D_x}(A_{x+t}-A_{\overset{1}{x+t.y}})$.

Write down the formula for the annual premium on the same on the supposition (1) that the premium is payable for the joint lives only, (2) during the whole continuance of the assurance.

Ans. The divisor for (1) is $1+a_{xy}$. For (2), it is

$$1+a_x-\frac{D_{x+t}}{D_x}(a_{x+t}-a_{x+t.y}).$$

14. Find the value of an assurance on the life of x provided he die AFTER y.

For which of the two lives would the usual medical examination be required?

Ans. (1) $A^2_{\overset{1}{xy}}=A_x-A_{\overset{1}{xy}}$. (2) The life y is the one which should be examined; but if y be much older than x, then x should also be examined.

15. Explain the construction and use of the various columns in a D and N table for two joint lives, including the columns for the calculation of contingent assurances.

16. Prove the formula for the value of an annuity payable half yearly,

$$\frac{a_x}{4}\left(2+\sqrt{v}+\frac{1}{\sqrt{v}}\right)+\frac{\sqrt{v}}{4};$$

and show that the common approximation, $a_x+\frac{1}{4}$, agrees with the proceeding to two decimal places, but not in the third place.

17. A person wishes to effect an insurance for the whole term of his life, in such a way that during the joint lives of himself and a much older person he may pay the smallest possible premium. How can this be arranged without prejudice to the Office granting the assurance?

Ans. It may be arranged that during the joint lives he shall pay a premium somewhat larger than the survivorship premium; and at the death of the elder life be allowed to continue the insurance at the ordinary rate of premium corresponding to his then age, independently of his state of health.

18. Find the annual premium for an insurance, on the supposition that interest at a given rate is allowed yearly to the assured on all premiums paid.

Ans. $\dfrac{M_x}{N_{x-1}-j(S_x+R_x)}=\dfrac{M_x}{N_{x-1}-jvS_{x-1}}$. Mr. Gray (on p. 64 of this volume) reduces this expression to $\dfrac{M_x}{R_x}$; but he omits to notice that this is only true on the supposition that interest is allowed to the assured *at the same rate as that by which the premiums are computed.*

19. Prove that the premium for a short term insurance on two joint lives is not much less than the sum of the premiums for insurances on the two lives separately.

What inference would you draw from this circumstance as to the value of a policy on two joint lives?

Ans. That the value of a joint life policy which has been in force for a few years, is small, as compared with the value of a policy on a single life.

20. Prove the formula for the value of an insurance payable at the instant of death:—

$$\frac{iA_x}{\log_e(1+i)}.$$

Expand this expression in a series as far as terms containing i^3.

Ans. $A_x\left(1+\dfrac{i}{2}-\dfrac{i^2}{12}+\dfrac{i^3}{24}\right)$.

21. Show how to find the value of the next presentation to a living.

22. Describe the two benefits of which an endowment assurance is composed, and from the annual premiums for those benefits deduce that of the endowment assurance.

Ans. These are a temporary assurance and an endowment, and adding the annual premiums for these two insurances, we get the annual premium for an endowment assurance equal to $\dfrac{M_x-M_{x+n}}{N_{x-1}}+\dfrac{D_{x+n}}{N_{x-1}}$.

23. The value of an annuity of £1 on the life of x, to commence on the death of y, may be expressed by the series

$$\frac{d_yN_x+d_{y+1}N_{x+1}+d_{y+2}N_{x+2}+\ldots\ldots}{l_yD_x}.$$

Explain the reasoning on which this formula is placed; and show its identity with the usual formula

$$\frac{N_x}{D_x}-\frac{N_{xy}}{D_{xy}}.$$

What is the signification of the first n terms of the series?

24. If P_x denote the annual premium for a whole life insurance, ${}_{\overline{n|}}P_x$ the annual premium for a term insurance for n years, and $\Pi_{x|n}$ the annual premium for an endowment payable at the end of n years, then the value of a policy effected at the age x, which has been n years in force, may be expressed

$$V_{x|n}=\frac{P_x-{}_{\overline{n|}}P_x}{\Pi_{x|n}}.$$

Establish this formula by direct reasoning, and show that it coincides with the ordinary formula.

25. There are n counters, marked respectively a, b, c, d, e, ... placed in a bag, and drawn out one at a time. Find the chance that neither a nor b nor c will hold its usual place among the letters.

Ans. $\dfrac{n^3-6n^2+14n-13}{n(n-1)(n-2)}$.

QUESTIONS FOR 1866.

1. Prove that the logarithm of a product is equal to the sum of the logarithms of the factors. In the common system of logarithms, what is the effect (1) of adding 1 to the logarithm, (2) of moving the decimal point one place to the right?

Ans. (1) We get the logarithm of ten times the given number. (2) We get the logarithm of the tenth power of the number.

2. Supposing a and b to be two large numbers, whose difference is small, prove the following formula for the logarithm of the intermediate number:—

$$\log_e \frac{a+b}{2}=\frac{\log_e a+\log_e b}{2}+\frac{(a-b)^2}{(a+b)^2+4ab}+\frac{1}{3}\left\{\frac{(a-b)^2}{(a+b)^2+4ab}\right\}^3+\ldots.$$

Adapt this formula to the common system of logarithms.

Ans. $$\log_{10}\frac{a+b}{2}=\frac{\log_{10} a+\log_{10} b}{2}$$
$$+\text{M}\left[\frac{(a-b)^2}{(a+b)^2+4ab}+\frac{1}{3}\left\{\frac{a-b)^2}{(a+b)^2+4ab}\right\}^3+\ldots.\right].$$

3. What is the probability that on an assigned Board day, the bank balance of an Insurance Company will be an exact number of pounds sterling?

What is the probability that the same will be the case at least once in the course of a year?

Ans. (1) $\dfrac{1}{240}$, (2) $1-\left(\dfrac{239}{240}\right)^{52}$.

4. There are two prizes in a lottery of 10,000 tickets. How many tickets must be taken to secure an even chance of obtaining a prize?

Ans. The number of tickets, x, is found from the equation $\dfrac{(n-x)(n-x-1)}{n(n-1)}=\dfrac{1}{2}$, and is equal to $n-\dfrac{1}{2}-\dfrac{1}{2}\sqrt{2n^2-2n+1}$; or $\left(n-\dfrac{1}{2}\right)\left(1-\dfrac{\sqrt{2}}{2}\right)$ approximately; or making $n=10{,}000$, $x=2929$.

5. Find the value of an annuity certain for n years, the first payment of which is to be made at the end of six months.

Ans. $\dfrac{\sqrt{1+i}}{i}\{1-(1+i)^{-n}\}$.

6. Describe the simplest practical method of computing a table of the present value of £1 due at the end of any number of years, at a given rate of interest.

7. Explain the method of forming a table of mortality from the death registers of a place, correcting for the increase of population.

8. Describe briefly the principles upon which a table of mortality is formed from the records of the transactions of a Life Insurance Company.

9. What are the principal mortality tables in use in England for life assurance purposes? and what are the chief arguments for and against the use of each?

Are the tables used for assurance calculations also suitable for the grant of annuities?

Ans. (1) "Northampton," "Carlisle," "Davies's Equitable," "Experience," "English Life." (2) Not generally, as it is important in the grant of annuities to distinguish between the sexes.

10. What is meant by the "expectation" or "mean duration" of a life of a given age?

Distinguish between this term, and the most probable duration of life; also the term which the life has an even chance of surviving.

11. Explain what is meant by the curtate expectation of two joint lives x and y. Prove that it is equal to

$$p_{xy,1}+p_{xy,2}+p_{xy,3}+\ldots.$$

12. Express the following quantities in terms of the original elements (v, l_x, l_y, l_{x+1}, &c.):—

$$\mathrm{N}_x,\ \mathrm{M}_x,\ \mathrm{R}_x-\mathrm{R}_{x+n},\ \mathrm{M}_{\overset{1}{\overline{xy}}};$$

and show that $\mathrm{M}_{\overset{1}{\overline{xy}}}+\mathrm{M}_{\overline{xy}}^{\ 1}=\mathrm{M}_{xy}$.

Ans. $$\mathrm{N}_x=l_{x+1}v^{x+1}+l_{x+2}v^{x+2}+l_{x+3}v^{x+3}+\ \ldots\ldots+l_sv^s$$

$$\mathrm{M}_x=(l_x-l_{x+1})v^{x+1}+(l_{x+1}-l_{x+2})v^{x+2}+\ldots+l_sv^{s+1}$$

$$\begin{aligned}\mathrm{R}_x-\mathrm{R}_{x+n}=&(l_x-l_{x+1})v^{x+1}+2(l_{x+1}-l_{x+2})v^{x+2}+3(l_{x+2}-l_{x+3})v^{x+3}+\ldots\\&+(n-1)(l_{x+n-2}-l_{x+n-1})v^{x+n-1}\\&+n\{(l_{x+n-1}-l_{x+n})v^{x+n}+(l_{x+n}-l_{x+n+1})v^{x+n+1}+\ldots+l_sv^{s+1}\}\end{aligned}$$

If $x-1>y$,

$$\begin{aligned}\mathrm{M}_{\overset{1}{\overline{xy}}}=&\tfrac{1}{2}(l_x-l_{x+1})(l_y+l_{y+1})v^{x+1}+\tfrac{1}{2}(l_{x+1}-l_{x+2})(l_{y+1}+l_{y+2})v^{x+2}+\ldots.\\&+\tfrac{1}{2}l_s(l_{s-x+y}+l_{s-x+y+1})v^{s+1}\end{aligned}$$

$$\begin{aligned}\mathrm{M}_{\overline{xy}}^{\ 1}=&\tfrac{1}{2}(l_y-l_{y+1})(l_x+l_{x+1})v^{x+1}+\tfrac{1}{2}(l_{y+1}-l_{y+2})(l_{x+1}+l_{x+2})v^{x+2}+\ldots.\\&+\tfrac{1}{2}(l_{s-x+y}-l_{s-x+y+1})l_sv^{s+1}\end{aligned}$$

13. Prove the following equation:—

$$a_{x-m\,\overline{m}|}\mathrm{A}_{x-n\,\overline{n}|}=a_{x-n\,\overline{n}|}\mathrm{A}_{x-m\,\overline{m}|}.$$

Ans. Each side is equal to $\dfrac{\mathrm{N}_x\mathrm{M}_x}{\mathrm{D}_{x-m}\mathrm{D}_{x-n}}$.

14. Given the values of the single and annual premiums for the insurance of £1 on a life x, find the rate of interest.

Ans. $$i=\frac{1}{\dfrac{\mathrm{A}_x}{\mathrm{P}_x(1-\mathrm{A}_x)}-1}.$$

15. Prove that

$$\frac{a_x a_{x+1} \ldots . a_{x+n-1}}{(1+a_{x+1})(1+a_{x+2}) \ldots . (1+a_{x+n})} = p_{x,n} v^n.$$

16. If P_x be the annual premium for the insurance of £1 on a life x, show that the half-yearly premium is nearly equal to $\dfrac{\frac{1}{2}P_x}{1-\frac{1}{4}(P_x+d)}$.

17. Give formulæ for the value of an endowment payable

(1) If x survive n years.
(2) „ both x and y survive n years.
(3) „ either x or y survive n years.
(4) „ x survive and y die within n years.

Ans. (1) $\dfrac{D_{x+n}}{D_x}$.

(2) $\dfrac{D_{x+n\, y+n}}{D_{xy}}$.

(3) $\dfrac{D_{x+n}}{D_x} + \dfrac{D_{y+n}}{D_y} - \dfrac{D_{x+n\, y+n}}{D_{xy}}$.

(4) $\dfrac{D_{x+n}}{D_x} - \dfrac{D_{x+n.y+n}}{D_{xy}}$.

18. Find the annual premium for a deferred annuity payable half-yearly; the first payment of the annuity being made six months after the payment of the last premium.

Ans. $\dfrac{N_{x+n}+\frac{1}{4}D_{x+n}}{N_{x-1}-N_{x+n}}$.

19. Find the value of a deferred annuity on the last of two lives.

Ans. $a_{\overline{xy}|\overline{n}} = a_{x|\overline{n}} + a_{y|\overline{n}} - a_{xy|\overline{n}}$.

20. Prove the formula for the annual premium for a temporary assurance, $v - \dfrac{N_x - N_{x+n}}{N_{x-1}-N_{x+n-1}}$.

21. Assuming that the deaths in each year of age are uniformly distributed, and supposing that two persons x and y both die in the same year, prove that the chance of x dying before y is accurately $\dfrac{1}{2}$.

22. Write down the three principal formulæ for finding the value of a policy.

Prove that the value is also equal to $1 - \dfrac{1-A_{x+n}}{1-A_x}$.

Ans. $V_{x|n} = A_{x+n} - P_x(1+a_{x+n})$
$= (P_{x+n} - P_x)(1+a_{x+n})$
$= 1 - \dfrac{1+a_{x+n}}{1+a_x}$.

23. Find the annual premium for the insurance of £1 on a given life, with return of all the premiums paid.

Ans. $\dfrac{M_x}{N_{x-1}-R_x}$.

24. It is required to effect an insurance on a given life in such a way that after n annual payments the premium shall be reduced one-half, after $2n$ payments it shall again be reduced one-half, and after $3n$ payments shall cease; determine the premiums.

Ans. The premium for the first n years

$$=\frac{4M_x}{4N_{x-1}-2N_{x+n-1}-N_{x+2n-1}-N_{x+3n-1}}.$$

25. If P_x denote the annual premium for an insurance on a life x, and $_{\overline{n|}}P_x$ the annual premium for a temporary assurance of n years on the same life, prove that

$$_{\overline{n|}}P_x=P_{x+n}-\frac{P_{x+n}-P_x}{1-(v-P_x)(v-P_{x+1})\ldots.(v-P_{x+n-1})}.$$

NOTES AND QUERIES.

Professor Oppermann, of Copenhagen, sends us the following curious problem:—

If x_1 be formed from x_0 by the formula, $x_1=x_0^{\frac{1}{x_0-1}}$

and in general, $x_{n+1}=x_n^{\frac{1}{x_n-1}}$

what will be the limit of the process?

He states that for every value of x_0, real or imaginary, except 0 or 1, $x_\infty=2$.

CORRESPONDENCE.

SOLUTION OF PROBLEM PROPOSED BY PROFESSOR DE MORGAN.

To the Editor of the Assurance Magazine.

Sir,—In the October number of the Assurance Magazine, Professor De Morgan, referring to Surrender Values of the form $1-\frac{1+a_y}{1+a_x}$ or $\frac{a_x-a_y}{1+a_x}$, says :—

"Now a_x-a_y is the value to (x) of a counter-survivorship—as we "may call it—of the following kind. The executors of the first who dies "pay an annuity of £1 to the survivor; and $(a_x-a_y)\div(1+a_x)$ is the "whole life premium which (x) should pay to be put in this position. "How, from the nature of this contract, does it follow that one payment of "this premium, over and above the annual premium which (x) should pay, "admits (y) to a policy of £1 at the premium for the age (x)?"

The following appears to me to be the reason.

Since a_x-a_y is the value of a counter-survivorship annuity of £1, $\frac{a_x-a_y}{1+a_x}$ will equal the value of a similar annuity of $\frac{1}{1+a_x}$.

It is required to prove that if there be paid to an Office a single premium of $\frac{a_x - a_y}{1+a_x}$, and an annual payment during y's life of $\frac{1}{1+a_x} - d$, (the premium which x would require to pay for an assurance of £1 or a perpetuity-due of d on his death), this will entitle y to an assurance of £1 or its equivalent, a perpetuity-due of d on his death.

If $\frac{a_x - a_y}{1+a_x}$ be paid, x will *receive* from the Office, should y die first, an annuity of $\frac{1}{1+a_x}$; and, should x die first, y will *pay* the Office an annuity of $\frac{1}{1+a_x}$; if $\frac{1}{1+a_x} - d$ be paid annually during the life of x, the Office will pay £1, or a perpetuity-due of d, on death of x.

Let us suppose that y dies first, x will receive, under the first contract, an annuity of $\frac{1}{1+a_x}$; but, as the annual premium $\frac{1}{1+a_x} - d$ does not cease till the death of x, x will require to pay this premium out of his annuity of $\frac{1}{1+a_x}$, thus reducing his annuity to d, since $\frac{1}{1+a_x} - \left(\frac{1}{1+a_x} - d\right) = d$. On the death of x his heirs will become entitled to the perpetuity-due of d which will then become payable under second contract. In effect, the single payment of $\frac{a_x - a_y}{1+a_x}$ and the payment of $\frac{1}{1+a_x} - d$ annually during the life of y, will provide an assurance of £1, or its equivalent, a perpetuity-due of d, on death of y should he die before x.

Next, let us suppose that x dies first, y will *pay* to the Office under the first contract an annuity of $\frac{1}{1+a_x}$, and will *receive* from the Office the perpetuity-due of d which will become payable under second contract through death of x. This reduces y's payment to the Office to $\frac{1}{1+a_x} - d$; and this payment will cease on the death of y, when his heirs will become entitled to the perpetuity-due of d. In effect, the single payment of $\frac{a_x - a_y}{1+a_x}$ and the payment of $\frac{1}{1+a_x} - d$ annually during the life of y, will provide an assurance of £1, or its equivalent, a perpetuity-due of d, on death of y should he survive x.

It will therefore be seen from the above that, whether y dies first or second, the single payment of $\frac{a_x - a_y}{1+a_x}$ and an annual payment during the life of y of the premium which a person aged x would require to pay for an assurance of £1, viz. $\frac{1}{1+a_x} - d$, will entitle y to a policy for £1, or its equivalent, a perpetuity-due of d, on death of y.

I remain,
Your obedient servant,
THOMAS MARR.

Glasgow, *24th October*, 1867.

ON THE AVERAGE AMOUNT OF A SUM INVESTED AT COMPOUND INTEREST FOR THE LIFE OF THE INVESTOR.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—"If in a group of individuals sufficiently large to insure an average mortality, each invests £1 at interest, what will the several sums amount to, *one with another*, at the end of the year of the death of each investor?"

Assuming *Annual* contributions, Mr. Morgan, under a very vague description computed a table for dealing with this question, of which Professor De Morgan gives a more lucid account in the *Companion to the Almanac* for 1842, reprinted in the *Journal*, vol. xiii., p. 141.

A leading Mutual Office having lately adopted the above principle in disposing of a portion of its ascertained profits, has drawn attention to the subject when viewed in its relation of an investment by *single* Premium, and I have taken the trouble of computing the examples annexed to show the probable result.

The function required for each age x is $(l_x-l_{x+1})(1+i)+(l_{x+1}-l_{x+2})(1+i)^2+(l_{x+2}-l_{x+3})(1+i)^3+\ldots$ &c., divided by l_x. Multiplying both numerator and denominator by $(1+i)^x$, and then summing the terms of the numerator as in the Annuity Col. N, we obtain a table of the ordinary Commutation form, from which the value for any age is found by division. The examples given below are based on the last (male) Government Table, taking interest at three per cent.

Age.	I. Assurance for £100 Premium.	II. Accumulation of £100.	III. Ratio of accumulation to Assurance Premium.
20	279·767	373·460	133·490
25	259·274	332·249	128·146
30	239·774	296·010	123·454
35	221·180	264·079	119·396
40	203·650	236·022	115·896
45	187·377	211·494	112·871
50	172·451	190·166	110·272
55	159·125	171·895	108·025
60	147·255	156·295	106·140

The table reads as follows:—

I. In a group of persons aged 20, each paying £100 into a common fund, on the condition that all should share in the proceeds alike (an ordinary Life Assurance), the representatives of each would receive at the end of the year of death £279·767.

II. Were the payments invested independently (at the same rate of interest) they would amount, *one with another*, at the end of the year of death, to £373·460.

III. The Single Premiums which would assure £100 payable to each of this age, would, if separately invested by the individual contributors, realize for them, one with another (being the average of the one year's amount receivable by some, and the much larger accumulation obtainable by a few) the sum of £133·490.

The last column affords by inspection facility for contrasting the benefits at different ages.

As the benefit contains that element of uncertainty which offers an attraction to many, it is matter of surprise that no other Office has before now hit upon this novel mode of dealing with its surplus.

I am, Sir,
Your most obedient servant,
H. AMBROSE SMITH.

5, Lothbury,
London, 7th December, 1867.

P.S.—It is worthy of note that the numerator of the first expression above, may be put into the following form:

$$l_x(1+i)+i\{l_{x+1}(1+i)+l_{x+2}(1+i)^2+l_{x+3}(1+i)^3+\ \ldots\ \&c.\}$$

This form is more elegant than the other, and being quite as easy of computation it might be used if we had to deal with only a single age. But it is unsuited for the formation of a Commutation Table as the terms do not represent *annual* values.

ON THE ADJUSTMENT OF PREMIUMS FOR LIFE ASSURANCE IN REFERENCE TO EXTRA RISKS.

To the Editor.

SIR,—There are two kinds of Extra Risk with which Assurance Companies in the ordinary transaction of their business have to deal, viz., that which arises from deteriorated health on the one hand, and that which consists in exposure to danger from some external cause, such as Climate, Military or Naval service, &c., on the other. The first risk is sometimes met by assuming an addition of a certain number of years to the actual age, and sometimes by a fixed addition to the annual premium, that is, fixed as regards the age and the nature of the assurance, but proportional to the sum assured. The second risk almost invariably by the latter method, except in those cases where the office is possessed of special Tables of Mortality founded upon observations made in the countries in which the parties reside, as India, for instance.

The adjustment of the premium by the former method,—that is, by a fixed addition to the age,—is open in some respects to considerable objection. For instance, if the life be young, and the assurance for a "Short Term," the addition of even several years would have comparatively but little effect upon the rate of premium. Nay, if Mr. Bailey's theory be correct, and it should ever be carried into practice, we should sometimes by this method obtain a *diminished* premium as a provision for a supposed *additional* risk! The second method, viz., that of a constant addition to the yearly premium irrespective of the age and the nature of the assurance, is not open to the particular objection just referred to,—but others, perhaps quite as strong, may be urged against it. In proof of this, it is only necessary to mention the case of "Endowment Assurances,"—to which indeed neither of the methods in question is applicable. There are two distinct benefits comprised in assurances of this description, viz., a Term Assurance and an Endowment; and the extra risk under the former is partly compensated by the diminished risk under the latter. The addition to the age which would be made if the case were that of an ordinary whole

life assurance,—with a corresponding postponement in the age at which the Endowment is supposed to become payable,—would not, in general, sufficiently provide for the extra risk;* while a constant addition to the premium, also corresponding to that which would be made on a whole life assurance, would be greater than the occasion demands,—inasmuch as it ignores the compensatory character of the contingency.

The practice of providing for the extra risk by a constant addition to the premium irrespective of the age, is evidently a sort of rule-of-thumb way of giving effect to the notion that the extra risk acts with equal force at all ages. We shall find reason to conclude that, as regards "Climate risks" at least, the idea in question is a singularly happy one. But whether true or not, if the hypothesis be adopted, there can be no good reason why we should not consistently act upon it, the more especially as our Life Annuity Tables, calculated at different rates of interest, afford us the means of doing so with perfect accuracy, and without any additional trouble whatever.

In a paper on "the law of mortality" read by me before the Institute of Actuaries on the 29th April last, and published in vol. 13 of this *Journal* (p. 325), I showed that if we have distinct Tables of the number of survivors at every age in two or more bodies of individuals subject to distinct causes of mortality, a Table may be formed showing the number of survivors in a body subject to *all* these causes, by simply multiplying the numbers at each age in the several Tables into each other. The demonstration of this theorem given in the paper above referred to has not to my knowledge been impugned, but in the discussion which followed the reading of the paper a very able member of the Institute, Mr. Sprague, expressed a wish for some further elucidation of a proposition which does perhaps at first sight seem somewhat paradoxical. Mr. Sprague probably thinks (as I do myself) that a ray of light is worth a bushel of demonstration; and I therefore make no apology for taking the first opportunity of complying with the suggestion.

Let A and A′ represent two individuals, each of whom is subject to the risk of dying from one of two distinct and independent causes. And suppose that the risk to which A is exposed arises from his being continuously fired at by a marksman, B; and that to which A′ is exposed arises in like manner from his being the target of another marksman, B′;—death from all other causes being in each case suspended. Now the first shot that takes effect, whether fired by B at A or by B′ at A′, will terminate the *joint* existence of the two lives. But it will evidently make no difference as regards the period at which the first shot takes effect if we suppose that B and B′ are both firing at A, instead of directing their fire at different individuals. Hence it follows, that the risk to an individual subject to two independent causes of mortality, acting simultaneously, is the same as the risk to the *joint* existence of two individuals, exposed, respectively, to one of those causes only; and if L_x and L'_x denote the survivors at age x under the latter supposition, then $L_xL'_x$ will correctly represent the survivors at the same age under the former.

This demonstration, if not more conclusive, is certainly more luminous than that given in the paper before referred to. I proceed now to apply the theorem to the matter which we have in hand.

Let the series L_x represent the survivors at successive ages in a body

* For the reasons previously stated it *might* produce a *diminished* premium.

subject to the *ordinary mortality only*, and L'_x those in a body subject to the *extra risk only*. Then if the extra risk be supposed constant we shall have $L_xL'_x=L_xk^x$; since the series L'_x will in that case form a geometrical progression. In the examples which follow I assume the ordinary mortality to be represented by the Carlisle Table,—the interest of money, by taking $v=(1\cdot04)^{-1}$,—and the extra mortality, by making $kv=(1\cdot06)^{-1}$ or $k=\frac{1\cdot04}{1\cdot06}=(1\cdot019)^{-1}$ nearly. The premium is in each case increased by a loading of 30 per cent.

I.—*Assurances without Extra Risk.*

Age.	Annual Premiums per Cent.			
	Assurances for the term of one Year.	Whole Life Assurances, Premiums payable during Life.	Whole Life Assurances, Premiums payable during Ten Years.	Endowment Assurances. (Age of 60.)
	£ s. d.	£ s. d.	£ s. d.	£ s. d.
20	0 17 8	1 14 3	4 1 1	2 4 2
30	1 5 3	2 5 8	5 0 10	3 5 4
40	1 12 7	3 1 9	6 4 9	5 6 8
50	1 13 7	4 7 6	7 13 7	11 9 1
60	4 3 10	7 3 10	10 12 2	

II.—*Assurances with Extra Risk.*

Age.	Annual Premiums per Cent.			
	Assurances for the term of one Year.	Whole Life Assurances, Premiums payable during Life.	Whole Life Assurances, Premiums payable during Ten Years.	Endowment Assurances. (Age of 60.)
	£ s. d.	£ s. d.	£ s. d.	£ s. d.
20	3 4 5	3 15 3	7 7 3	4 1 11
30	3 11 11	4 5 6	8 0 1	5 0 5
40	3 19 1	5 0 0	8 16 3	6 18 2
50	4 0 2	6 3 7	9 16 2	12 14 10
60	6 9 4	8 19 6	12 8 2	

III.—*Difference, or true Extra Premium.*

Age.	Annual Extra Premiums per Cent.			
	Assurances for the term of one Year.	Whole Life Assurances, Premiums payable during Life.	Whole Life Assurances, Premiums payable during Ten Years.	Endowment Assurances. (Age of 60.)
	£ s. d.	£ s. d.	£ s. d.	£ s. d.
20	2 6 9	2 1 0	3 6 2	1 17 9
30	2 6 8	1 19 10	2 19 3	1 15 1
40	2 6 6	1 18 3	2 11 6	1 11 6
50	2 6 7	1 16 1	2 2 7	1 5 9
60	2 5 6	1 15 8	1 16 0	

These Tables show the effect of a supposed constant extra risk upon the Annual Premium required for different kinds of assurances on single lives;

from an inspection of which we may judge how far a uniform extra premium is calculated to provide for it. I now give the formulæ used in their computation,—deferring for another occasion the examination of cases involving two or more lives.

In each of the foregoing examples $D_x = L_x \times 1{\cdot}04^{-x}$. Let D'_x denote $L_x \times 1{\cdot}06^{-x}$, and $N'_x = D'_{x+1} + D'_{x+2} + \ldots$ We shall then have:

1. Premium for an assurance for the term of one year

$$1{\cdot}04^{-1} - \frac{D'_{x+1}}{D'_x}.$$

2. Annual Premium, payable during life, for an assurance for the whole term of life

$$\frac{D'_x}{N'_{x-1}} - (1 - 1{\cdot}04^{-1}).$$

3. Annual Premium, payable during 10 years, for an assurance for the whole term of life

$$\frac{D'_x - N'_{x-1}(1 - 1{\cdot}04^{-1})}{N'_{x-1} - N'_{x+9}}.$$

4. Annual Premium, payable during $60-x$ years, for an Endowment Assurance payable at sixty or death.

$$\frac{D'_x}{N'_{x-1} - N'_{59}} - (1 - 1{\cdot}04^{-1}).$$

The formulæ for other cases may be deduced with equal facility. That for a single premium for a whole life assurance would be

$$1 - \frac{N'_{x-1}}{D'_x}(1 - 1{\cdot}04^{-1}).$$

I do not know how this case would be treated by the rule-of-thumb method; but if any courageous advocate of that system should choose to take up the cudgels for it, we shall doubtless be informed upon this point.

Having seen how the hypothesis of a constant extra force of mortality operates upon the calculation of premiums for different kinds of assurance, there remains now to consider the case of a transfer from one condition to the other; the most frequent and striking instance of which is that of an assurance effected on a life resident in India, who returns after the lapse of many years to reside permanently in Europe.

It is not surprising that the practice of charging a constant extra premium for all ages and for all kinds of assurance should have been accompanied by the corresponding primitive one of simply discontinuing it when the extra risk had ceased. This, I imagine, has led to the curious but very general practice which still obtains in Indian Offices, whose rates are now, and have long been, computed by mortality tables formed exclusively upon observations made in India, viz., to fix the European rate, not by a calculation based upon the actual age at the time of arrival in Europe, but according to the European risk for the age at which the assurance was effected. A practice so utterly void of any rational foundation could only

have originated in the very infancy of the science of actuarial computation, and it is little to the credit of the profession that it should have maintained its ground to the present day.

The true mode of procedure presents no difficulty whatever, notwithstanding the formidable array of "Gorgons, and Hydras, and Chimeras dire," which a writer on this subject, in a former volume of the *Journal*, so unnecessarily conjured up in his path; the principle to be kept in view being simply that the estimated liability of the Office, under its contract, shall be the same after the change has been effected as it is at the time the application for the transfer is made. This is the utmost that the most unreasonable Policyholder can possibly expect, and it is no more than the most prudent Office can with perfect safety concede.

To obviate any question as to the mode of adjusting the loading (which would open up another and very wide field of inquiry), I will suppose the assurances to be affected at pure premiums. Let a_x represent the value of an annuity at the ordinary (or European) risk, and a'_x the same at the increased (or Indian) risk. To determine p the future reduced premium we have the following equation of condition:

$$1-\frac{1+a'_{x+n}}{1+a'_x}=\{1-(1-v)(1+a_{x+n})\}-p(1+a_{x+n})$$

whence we obtain

$$p=\frac{1+a'_{x+n}}{(1+a'_x)(1+a_{x+n})}-(1-v).$$

This formula enables us to see at a glance the relation which the reduced premium bears, 1st to the premium for the original age by the Table on which the assurance was effected, 2ndly to the premium for the original age according to the Table for the reduced risk, and 3rdly to the premium for the reduced risk for the present or actual age of the life assured. For the first is:

$$\frac{1}{1+a'_x}-(1-v),\quad \text{while } p=\frac{1}{1+a'_x}\cdot\frac{1+a'_{x+n}}{1+a_{x+n}}-(1-v).$$

The second is:

$$\frac{1}{1+a_x}-(1-v),\quad \text{while } p=\frac{1}{1+a_x}\cdot\frac{(1+a_x)(1+a'_{x+n})}{(1+a'_x)(1+a_{x+n})}-(1-v).$$

And the third is:

$$\frac{1}{1+a_{x+n}}-(1-v),\quad \text{while } p=\frac{1}{1+a_{x+n}}\cdot\frac{1+a'_{x+n}}{1+a'_x}-(1-v).$$

Hence we draw the following conclusions, viz.:

1. If $1+a'_{x+n}=1+a_{x+n}$, or the annuities for the actual age by the two Tables are equal in value (which might easily be the case coexistently with very different values at earlier ages) no reduction would take place in the premium. This result is evidently perfectly consistent with the supposition.

2. If $\frac{1+a_x}{1+a'_x}\cdot\frac{1+a'_{x+n}}{1+a_{x+n}}=1$ or $\frac{1+a_x}{1+a'_x}=\frac{1+a_{x+n}}{1+a'_{x+n}}$, that is, if the values of annuities (in advance) by the two Tables were in a constant proportion

to each other,—the present practice of reducing to the European rate for the original age would be correct.

3. If $1+a'_{x+n}=1+a'_x$,—that is if the value of the annuity on the life (at the increased risk) for the original age were the same as that for the present or actual age,—the reduced rate would be that required for a new assurance. This is precisely what might have been expected, for the supposition implies that the risk had not increased since the commencement of the assurance, and therefore that the Policy had acquired no value.

Mr. Neison's observations on the Bengal Military Mortality are now generally accepted as the best index of the value of European life in India. In the following Table, therefore, designed to test the correctness of the hypothetical equation

$$\frac{1+a'_x}{1+a_x}=\frac{1+a'_{x+n}}{1+a_{x+n}}$$

(which we have seen is virtually assumed in the prevailing practice in determining the reduction of the premium) I have given the results derived from those observations in juxta-position with the results of the supposition of a constant additional force of mortality. The values contained in the first three columns are those of annuities in advance.

Age.	(1.) Carlisle 4 per Cent.	(2.) Carlisle 6 per Cent.	(3.) Bengal M. 4 per Cent.	(4.) Ratio of (2) to (1).	(5.) Ratio of (3) to (1).
20	19·362	14·835	15·126	·766	·781
30	17·852	14·020	14·283	·785	·800
40	16·074	13·002	13·201	·809	·821
50	13·869	11·631	11·773	·839	·849
60	10·663	9·304	9·849	·873	·924

These results show us that whether we proceed upon the theory of a constant additional force of mortality, or whether we take the actual mortality among Europeans resident in India, the values of annuities (in advance), with and without extra risk, when computed at the rate of interest most commonly realized, tend to a ratio of equality as the age increases; and hence we infer (from the first of the conclusions drawn from an examination of the formula for the reduced premium) that the reduction to the rate for the original age is greater than the results of computation will warrant.

Not to trespass too much on the patience of your readers, I will now bring my letter to a close, with the intention of resuming the subject in another communication.

I remain, Sir,

Your very obedient servant,

W. M. MAKEHAM.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

Report on the Sixth International Statistical Congress. *By* Samuel Brown, Esq., F.S.S., *President of the Institute of Actuaries.*

[Read before the Institute, 25th November, 1867.]

THE studies of an actuary depend so much for their success on the accurate and full collection of statistics, that we cannot but feel a deep interest in every effort that is made to improve the method of and to lay down sound principles for obtaining them. Hitherto we have been more especially concerned with the laws of population, vitality, and disease, because they form the basis of the vast extension of life assurance in this and other countries. But in the course of the last few years the progress of social science has brought forward many questions to which the doctrine of probability as to the recurrence of events may be applied, which, though not strictly within the limits of our professional pursuits, afford many opportunities for the use and extension of our science.

The Statistical Congresses, of which six have now been held, have been fruitful in results of the highest utility. The object of them has hitherto been to agree upon some uniform method of collecting the facts which show the condition and progress of a nation, whether in its social, commercial, or economical relations, which will enable the true laws to be discovered by which such facts are found regularly to recur under similar conditions.

It is easy to see that no private individuals could have power or authority sufficient either to collect such statistics on the proper scale or to prescribe the mode and forms of the schedules to be used for the purpose. It must be especially the work of Government, using the authority and resources of the nation, to make through its officials the needful researches. But previous to the formation of these congresses there was no similarity either in the division of the subjects or in the form of the documents. It was very seldom possible to compare together the results of an enquiry into the same subject in two different countries. Yet it is essential, if we wish to obtain the law which governs a certain class of events, to study its operations under different conditions and in different countries.

To take the subject with which in this Institute we are most familiar—the law of population—it will recur to every one that a few years back the statistics of mortality in different countries were scarcely ever given for the same combination of ages, nor were the censuses taken at the same interval of time. In the schedules of the census the same facts were not asked for. It was impossible to compare the emigrations from one state with the arrivals in another. The divisions of labour and of occupation were wholly different, and so many other points were left in obscurity or presented in such different ways that very little was gained by endeavouring to trace the causes of growth or decrease of population by bringing together the statistics of different countries. But within the last fifteen years a very great improvement is perceptible in the population returns of all European countries, and though much remains to be done to produce uniformity, the effect of these periodical meetings is seen in every new census which appears.

It is principally to M. Quetelet, to whose writings, eminent position, and incessant labours the study of statistics owes already the position of a science, that we are in a great measure indebted for the success of these congresses. The first, which was held in Brussels in 1853 under his auspices, formed the model of the rest. Called by the Government, who first sent invitations to the Governments of other countries, its most important feature was then and always will be the part taken by the official delegates in discussing the resolutions and seeing them carried into effect. But it was then wisely determined to include in the invitations a considerable number of scientific men, or those who had studied the special subjects to be discussed, so as not only to secure the actual practical effect of the conclusions arrived at, but to get them

recommended as based upon the largest possible experience, and, in some points at least, the agreement of men of the most divergent views.

The second congress was held in Paris in 1855, the third in Vienna in 1857, the fourth in London in 1860, and the fifth in Berlin in 1863.

At the close of the latter congress, several invitations were presented by the delegates in the names of their Governments for the next meeting, and it was left to the organization commission to decide the question, which they did in favour of Italy, and the city of Florence. A commission comprizing some of the most eminent Italian statesmen and men of science was thereupon nominated under the presidency of the Minister of Agriculture, Industry, and Commerce; and the King, Victor Emmanuel II., to mark the interest which he took in the successful progress of statistics, appointed his eldest son, Prince Humbert, of Savoy, President of the Congress.

The great political events of 1866 deferred the meeting till the present year, when its success was marked by having a larger number of members than any of the previous congresses. No less than 632 natives, and 85 foreigners, total 717, attended. In London, which was the next largest meeting, there were 505 natives and 90 foreigners, including in the former, the representatives of our colonies, total 595.

The Minister of Agriculture, Industry, and Commerce opened the commission by a lucid address on the subjects to be studied, the necessity of clearness and precision in stating them, and the aim and object of obtaining uniformity in statistical returns.

Dr. Pietro Maestri, the director of the Royal Statistical Bureau of Italy, whose able writings have made him honourably known throughout Europe, was indefatigable both by his reports and personal labours in preparing the programme for discussion, and he contributed largely to the success of the congress.

The question of the programme was rather a difficult one. In the five previous meetings so many subjects of public utility had been discussed and settled by resolutions, that he wisely advised keeping in a great measure to those questions which, having been only slightly debated or deferred for further consideration, left a fair legacy of work from previous years. To these, however, he added various questions which had been proposed in a special correspondence with the chiefs of the statistical departments of other Governments, and which involve some of the most important

questions of the day, such as agricultural statistics, money and financial credit, the joint publication of statistics of all countries, and their transmission free of expense between the Governments, physical and moral statistics, interior commerce, improvement in the registers of population, &c.

Besides these useful reports Dr. Maestri undertook a most important labour in carrying on an admirable work prepared by Dr. Engel for the congress at Berlin, being an index and summary of all the matters treated of and resolutions passed at all the previous congresses. From the great variety of matter and numerous subdivisions of each subject it would be impossible to avoid useless repetitions of some and serious omission of others; unless a good index could be formed of all that has been done as well as of what has been commenced but left unfinished. Dr. Maestri has merely extended Dr. Engel's plan, so as to include the resolutions and labours of the Berlin congress; but it may be imagined what suggestive matter it contains and what a variety of details must be contained in the original programmes of each congress when a mere index to five of them occupies a thick quarto volume of 330 pages. It includes also a synopsis of the reports presented by the official delegates on the condition of statistics and statistical studies in their respective countries, as well as of the classes of the members and of the literary works that were presented at each congress.

With these materials for preparing the programme, the committee appointed for the purpose was divided into eight sections, to which the following subjects were confided:—

Section I.—Theory and practice of statistics.

1. Re-organization of the International Congress.
2. Constitution of official statistics.
3. Legal population of states—*i.e.*, of natives absent from the country as well as those present at the time of the census.
4. Laws of mortality and normal tables for assurance societies.
5. Uniform nomenclature for statistics.

Section II.—Topography.

1. Organization of meteorological stations and formation of a daily chart for all Europe.
2. Nature, ownership, and regulations for the supply and use of water. Water for drinking, for irrigation.

Section III.—Agrarian statistics.

1. Valuation of the net revenue of cultivated lands and value of products.
2. Economy of the Credit Foncier (land loans).
3. Statistics of cattle—production, importation, exportation.

Section IV.—Statistics of communes.

1. Constitution as to boundaries and economical condition of communes.

Section V.—Statistics of monetary circulation and paper currency.

Section VI.—Moral and judicial statistics.

1. The poor classes—mendicants in the streets and at church doors, people admitted into workhouses, nocturnal and similar asylums, tramps, juvenile offenders, liberated convicts, prostitutes.
2. Uniform rules for collecting in the different countries of Europe statistics of the domestic relations of families (such as guardianships, divorces, &c.)
3. Statistics of bankruptcies and the effects of different systems of legislation on commercial credit.
4. Statistics of restraint of personal liberty in civil and commercial matters.
5. Statistics of the causes of crime.
6 Statistics of offences against military and naval discipline and their punishments, in order to be able to form a comparison of the moral condition and discipline of the various armies and naval forces of different countries, and of the efficacy of the means of repression used.

Section VII.—Military statistics.

1. Health and mortality of the civil and military population.
2. Enquiries as to the food, dress, equipment, lodging, and service of the military and naval forces.
3. Gymnastic exercises.
4. Forms of schedules concerning the physical condition, invaliding, and mortality of the military and naval services.
5. Classes of disease in reference to duration of service.

Section VIII.—Education.

1. Schools of fine art, museums, archives, or collections of public documents, libraries.

Reports upon all these subjects so important and of such varied interest, were prepared by different members of the eight sections

to constitute the total programme. Amongst the writers will be found the names of some of the most eminent authors and thinkers, whether Italians or other members of the congress. In the first section are found Messrs. Visschers, Berg, Correnti, and Anziani, Brioschi and Maestri. In the second Professor Cantoni wrote on meteorology; and a long and very elaborate report on hydrography appears by S. Pareto. Agricultural statistics employed the pens of S.S. Rabbini, Restelli, and Lampertico. Currency and money and unity of weights and measures were treated in an able report by S. Allievi. The subjects of the sixth section were discussed by Dr. Maestri and Professor Messedaglia, of the seventh by S. Baroffio, and the greater part of the eighth by the indefatigable Dr. Maestri and others.

I do not propose to give an account of the discussions on this wide range of subjects, however important most of them are to every man who is watching the great social questions of the day, bnt briefly to describe the method of proceeding, and state the conclusions arrived at on one or two of the questions which are likely more especially to interest the members of this institute.

For two days before the Congress, preliminary meetings of the foreign delegates were held under the presidency of M. Quetelet, at which the question of the free transmission of official statistical documents was referred to a special committee. Resolutions were also passed in favour of a cheap annual *resumé* in each country of its principal official statistics, with a comparison of the same in previous years somewhat in the form of the English "Statistical Abstract;" also of publishing an annual index with details sufficient to give a clear idea of the contents of all the documents printed by the Government during the year.

The Congress was formally opened on the 29th of September by a short address from the Minister of Agriculture, S. De Blasüs, in which, after alluding to the ancient renown of Italy in learning and the arts and the wide field for the cultivation of statistical studies which the recent union of nearly all its provinces into one opened up, he welcomed in cordial terms the many distinguished foreigners in the name of the Government and Italy itself. The Government delegates of foreign countries were then elected vice-presidents, several secretaries were appointed for different languages, though it was understood that Italian and French were to be generally used in the public discussions, and the members separated into the different sections to discuss the parts of the programme allotted to each.

At the second session on the following day, the Minister of Agriculture in a long discourse pointed out how the reforms in all branches of the Italian administration had led to the appointment of Parliamentary commissions for statistical researches in every state question, of the results of which he had ordered a complete collection to be laid before the Congress. He pointed out with justice how liable statistics are to be collected in aid of a particular interest or prejudiced view, instead of being based upon general and sound principles. In the annual and quarterly Government publications will be found most striking details of the effect of improvements in Customs, laws, in railways and roads, telegraphs, and other means of social progress in Italy. He concluded by showing the immense advantages attending uniformity of observation, so as to compare the same classes of facts amongst different nations, and the great benefits which a congress moving about for this object is likely to confer on all countries where it assembles.

Amongst the subjects discussed in the sections was a report by S. Allievi on uniformity of weights and measures in all countries, on which the resolutions introduced by an eloquent speech by M. Wolowski, a distinguished member of the Institute of France, were carried unanimously. They were to the effect that this Congress, confirming what had passed in all the previous congresses, recommends the universal adoption of the metric system; that to aid its progress associations should be formed in each country where it is not yet adopted, to make known its advantages; that the International Decimal Association in London should, from correspondence with these societies, obtain the materials for a report to be made at the next Congress on the advances made and the difficulties which have been or have to be overcome in its progress; that it should form in all countries a part of the instruction in Government and other schools, and that the other means recommended in M. Jacobi's report should be adopted for its extension. However much patriotic motives may encourage the adoption of an isolated system for each country, there can be no doubt of the immense facilities which a uniform decimal system would give to commerce and national intercourse, and none who have studied the subject will deny the extreme simplicity of the metric system in use, and the time and mental labour it would save in the education of youth. Weight and measure are surely cosmopolitan. Why should the instruments for computing them make them merely local?

A further proposition on this subject was adopted at the sixth

session, that the chiefs of departments should prepare for the next congress a table of the weights, measures, and coins used in their own country, with their reduction into those of the other principal states.

As to the almost equally important but more difficult subject of international money, the Congress confined itself to approving the adoption of any suitable means to extend the principles of the Monetary Convention of Paris, 23rd December, 1865. In the meantime it was decided to recommend to the official departments of states to collect all the statistics of gold production and of coinage in uniform schedules.

Passing over the resolutions on hydrography introduced by S. Pareto; on agricultural statistics, especially the supply of animal food, by S. Lampertico; of meteorology, by S. Cantoni, and the other subjects referred to the sections, all of which gave rise to animated discussions, but which were finally adopted, and an important discussion on the permanent organization of the congress, which ended in deferring the whole question till a further congress should think fit to resume it, I draw attention to a very admirable work presented by M. Quetelet. This was a publication of statistics relating to the population of different states. With the co-operation of the Directors of the Statistical Departments of other Governments, and by the efficient aid of M. Heuschling, of the Statistical Commission of Belgium, it comprised the principal tables of population, deaths, births and marriages brought together as nearly as possible to the same period, about the year 1861. It was the result of a proposition made by M. Quetelet himself to the congress at London, and had been printed at the expense of the Belgian Government. Though considered only as an essay, it had been so favourably received that he proposed to continue a similar work in other branches of enquiry. It must be admitted that this is indeed a most important beginning, not merely because it contains the summaries of several large volumes in a form suitable for computing the percentages for comparison, but because the information may be accepted as the most authentic that can be obtained. It is to be hoped that M. Quetelet will be able very soon to extend his plan, especially to the statistics of commerce, law, crime, and finance.

Another most important proposition was also made by the same eminent authority, and unanimously adopted by the first section, and afterwards by the Congress, that "considering the importance and extension of statistical questions to which mathe-

matics may be applied, and that in all civilized nations some illustrious geometricians have made the application of the doctrines of probability to such subjects their special study there should at the future congresses be a particular section to consider statistical questions in their direct relation with the theory of probabilities." I need hardly point out to the members of this Institute how clearly this resolution recognizes our own objects and pursuits, and how much we owe one of the most eminent of our honorary members for the high position which he thus incidentally claims for us.

In regard to other questions of population, the seventh section, which was presided over by Dr. Graham Balfour, sent in propositions as to the sanitary condition of military and naval forces, including enquiries into their diseases and mortality, according to locality, age, and duration of service, medical treatment and surgical operations, the care of the wounded in time of war, &c. From the great value of the reports on the foreign military stations of the British army, which for so many years have been prepared by the care and talent of Dr. Graham Balfour, it is easy to perceive what great additions to our knowledge of the effects of climate on health would be gained by a comparison of such statistics in every country.

The resolutions on the subject of laws of mortality and normal tables for assurance societies introduced by S. Brioschi in a very interesting memoir in the programme were carried, thus:—

1. That the Congress considers it desirable that each government should publish normal tables corresponding to the operations of the different classes of life assurance societies, which being corrected and re-published whenever the original mortality tables are revised, may serve as guarantees of safety to the public.

2. That the Governments should obtain the experience of the assurance companies so far as concerns the mortality amongst their members, and publish the results.

In the debate on this subject S. William Rey, director of the Reale Life Assurance Company at Milan, and who has recently published a new and valuable series of Mortality Tables for North Italy, graduated according to the Gompertz theory, made a very effective speech.

It will be observed that these proposals do not involve the large and very difficult question, whether it is to the advantage of the public that Government should undertake assurance business generally, and as the publication of any information that can be

obtained must benefit science, it is to be hoped that these resolutions will be practically carried into effect.

Some general propositions to obtain a common language for statistics, that is, that in all publications the same terms should have the same definite meanings, were passed, and it was referred to the chiefs of statistical departments in each country, to consult and agree upon a uniform nomenclature, and in all documents to define the old as well as the newly recognized meaning of the term. Anyone, for instance, who has had to study criminal statistics will admit the great difficulty of comparing the statistics of crimes and punishments from the very different meaning which the same word conveys, not merely in different countries, but in the different provinces and divisions of the same country. This is to cite but a very small part of the difficulties of the subject, but it shows the absolute necessity of agreeing in all cases to some terms conveying the same and definite meanings. At the previous congresses, great progress had been made in a similar process by agreeing to names common to all countries under which the causes of death in the tables of mortality may be classified. In this useful labour Dr. Farr has taken a very prominent part.

It would occupy too much time to detail all the other resolutions or to give a *resumé* of the debates. I have only stated those which appeared to me most likely to attract your attention. After a week's sittings the congress was closed by a short address from his Excellency the Minister of Agriculture, S. De Blasüs, in which he gracefully alluded to the great advance made by the labours of the members and his hope that whilst Italy would profit by the new knowledge she had gained, the distinguished strangers on their return would bear a pleasing remembrance of the meeting and of the earnestness with which Italy was entering upon the highest studies of practical life and of strict intellectual truth.

Indeed every one there present must feel that a tribute is due to the hospitality and courtesy which they received from all sides. His Majesty the King of Italy received the foreign delegates at the Pitti Palace, and after apologizing for the absence of his eldest son, the Prince Humbert, who was detained by urgent affairs at Paris, invited them all to a grand banquet which was given at the close of the congress. The President of the Ministerial Council, S. Ratazzi, and the Minister of Agriculture, gave a grand reception in the ancient and historical Palazzo di Podesta, now converted into a national museum of rare antiquities. The syndic and municipality of Florence invited the members to the municipal

palace on the Cascine and to a musical entertainment in La Pergola, and the Marquis di Pepoli, as syndic of Bologna, offered them the hospitality of that renowned city. From all classes, eminent statesmen and learned professors from all parts of Italy, and above all from Dr. Maestri, the chief of the Statistical Department, and to whom the congress owes so much for its success, the most courteous and hearty attentions were received.

I will conclude by observing that, though our mere professional pursuits only comprise a small part of the varied and most important topics to which I have alluded, the same theory of probabilities on which they are based is capable of application to many of the others. Knowledge always repays the toil by the pleasure itself of its acquisition. Still more must this be the case when the mind is given to subjects in which the health and happiness of thousands is involved and the progress of civilization advanced. If, therefore, when the members of this Institute have thoroughly grounded themselves in the knowledge of their profession, they should be disposed to extend the application of their science to some of the questions of which such a wide field of view is here opened up, they will have the satisfaction of knowing that in abolishing error, prejudice, or ignorance, they are using the basis of their professional skill for the good of their own community and the social progress of nations.

On the final law of the sums of drawings. *By* A. De Morgan, Esq.

LET there be letters $x, y, z, \ldots$ each of which has values, choices, or drawings. Let their number be σ, and let, $\xi, \eta, \zeta, \ldots$ be their several numbers of drawings: $x_1, x_2, \ldots x_\xi$ the drawings of x; and so on. Let $\Sigma x, \Sigma y, \ldots$ be the sums of the drawings of x, y, &c.; and $\Sigma : x$, $\Sigma : y$, the average drawings: so that $\Sigma : x = \frac{1}{\xi} \Sigma x$, and so on. Let any drawing of one letter be compatible with any drawings of the others: so that the terms of the product $\Sigma x . \Sigma y . \Sigma z$ contain every joint drawing of xyz. The number of drawings of xyz is $\xi\eta\zeta$: and $\Sigma(xyz) = \Sigma x . \Sigma y . \Sigma z$. Dividing both sides by $\xi\eta\zeta$, we have $\Sigma:(xyz) = \Sigma:x . \Sigma:y . \Sigma:z$. The same holds for any number of letters, or powers of letters: and thus we have the following theorem:—All possible combinations of drawings being taken into account, the average of all the

drawings of a product is the ***similarly* resolved*** product of the averages of all the drawings of the factors. The sum $x+y+z+\ldots$ has $\xi\eta\zeta\ldots$ drawings, say N. For a given set of drawings x_a, y_b, z_c, of a given number of letters, the expression $x_a+y_b+z_c+v+w\ldots+$ has $\mathrm{N}:\xi\eta\zeta$ drawings. Consequently the number of drawings of $(x+y+z+v+w+\ldots)^k$ in which $\mathrm{P}x_a^\alpha y_b^\beta z_c^\gamma$ occurs $(\alpha+\beta+\gamma=k)$ contributes $\mathrm{NP}x_a^\alpha y_b^\beta z_c^\gamma:\xi\eta\zeta$ to $\Sigma(x+y+z+\ldots)^k$, and, dividing by N, contributes $\mathrm{P}x_a^\alpha y_b^\beta z_c^\gamma:\xi\eta\zeta$ to the average. But this, by the preceding theorem, when a sum is made from all values of a, b, c, is P multiplied by the average of the product $x^\alpha y^\beta z^\gamma$. Hence this theorem;—The average of the drawings of $(x+y+z+\ldots)^k$ is the sum of the averages of the several terms in the multinomial development of $(x+y+z+\ldots)^k$.

Next, let h be the number of letters in a certain term; four, for instance, in $\mathrm{P}x^\alpha y^\beta z^\gamma v^\delta$. The whole number of letters being σ, the number of such terms, no two of which have the same letters, is h_σ, the number of combinations of h out of σ. If we take one such term, and if all the letters $\alpha, \beta, \gamma, \ldots$ be different, the number of terms containing these letters is the number of arrangements of h, or $h(h-1)\ldots 3.2.1$, say $[h$. But if there be repetitions, say for instance α occurs a times, β b times, &c., so that $\mathrm{a}\alpha+\mathrm{b}\beta+\mathrm{c}\gamma+\ldots=k$, $\mathrm{a}+\mathrm{b}+\mathrm{c}+\ldots=h$, then the number of terms in which one set of letters occurs is the number of distinct ways of distributing a marked α, b marked β, &c., in $\mathrm{a}+\mathrm{b}+\mathrm{c}+\ldots$ places: or $[h\div[\mathrm{a}.[\mathrm{b}.[\mathrm{c}\ldots.$ Hence the total number of terms of the type $\mathrm{P}x^\alpha y^\beta z^\gamma\ldots$ is h multiplied by the preceding or $[h.\sigma(\sigma-1)\ldots.(\sigma-h+1)$ divided by $[h.[\mathrm{a}.[\mathrm{b}.[\mathrm{c}\ldots.$ If σ increase without limit, this approaches to ratio of equality with $\sigma^h\div[\mathrm{a}.[\mathrm{b}.[\mathrm{c}\ldots$ and this we may write as the infinite number of terms of the type given, when the number of letters is infinite. In this case, the type having the greatest value of h gives a result infinitely greater than all the others put together, *if the drawings be all positive,* which at first we shall suppose. The degree of the term being uniform $(=k)$, h is greatest (and $=k$) when all the exponents α, β, &c., are severally equal to unity. The multinomial coefficient of such a term is $[k$. Hence, σ being infinite, $(x+y+z+\ldots)^k$ is $[k\times$ the sum of all products of $x, y, \ldots k$ and k together: or the rejected part is an infinitely small fraction of the retained part. Hence, by the preceding theorem, we find that

* Thus Σx^2 is not $\Sigma x.\Sigma x$, if ξ drawings of x^2 be found from ξ drawings of x. But Σxx *is* $\Sigma x.\Sigma x$, if ξ^2 drawings of xx be made by combining ξ drawings of the first x with ξ drawings of the second.

the average of $(x+y+\ldots)^k$ is [$k\times$ the sum of all products of $\Sigma:x$, $\Sigma:y$, &c. k and k together. But if we repeat this reasoning upon $(\Sigma:x+\Sigma:y+\ldots)^k$ we find the same [$k\times$ sum of combinations of $\Sigma:x$, $\Sigma:y$, &c. k and k together, for the value of this new multinomial power. Consequently

$$\Sigma:(x+y+\ldots)^k=(\Sigma:x+\Sigma:y+\ldots)^k$$

or:—All drawings being positive, and the number of letters infinite, the average drawing of the kth power of the sum is a *subequal** of the power of the sum of the several average drawings.

All the theorems which come under this subject have a remarkable quality. By the look of the demonstration, it should seem as if we must take a large number of letters to have a chance of even a glimpse of verification. It is not so: we may get a good glimpse out of three letters. Let each letter have drawings 1, 3; and take $x+y$. Its drawings of $x+y$ are 2, 4, 4, 6; the sum of averages is $2+2$. The averages of the sums of the 1st, 2nd, 3rd, 4th, powers of 2, 4, 4, 6 are 4, 18, 88, 456; of which it can only be said that the powers of $2+2$ are not forcibly suggested. Take three letters: the values of $x+y+z$ are 3, 5, 5, 7, 5, 7, 7, 9; and the average sums of powers 6, 39, 270, 1965; these much more nearly suggest the powers of $2+2+2$.

I now come to cases of both positive and negative drawing. But the drawings are to be *balanced:* that is, for every positive drawing the corresponding negative drawing exists; and *vice versâ.* Some very appropriately call such things *plusminus* drawings. Thus if 7, 7, 7 occur among the drawings of x, so do -7, -7, -7. Very slight attention will now make it apparent that in $\Sigma(x+y+\ldots)^k$ every term $Px^\alpha y^\beta\ldots$ in which one or more of $\alpha, \beta, \ldots$ are odd gives a set of drawings the sum of which vanishes. And as this must happen in *every* term when k is odd (since $\alpha+\beta+\ldots=k$) it follows that $\Sigma(x+y+\ldots)^k$ is always $=0$, when k is odd. We proceed to consider even powers, as in $(x+y+\ldots)^{2k}$. The reasoning of the preceding case, if understood, will lead us immediately to a new result. All cases being disposed of as evanescent in which one or more exponents are odd, the terms of the form $Px^2y^2\ldots$ having k letters in each, give a sum infinitely above that

* I use this word, which I find more and more convenient, to denote equality within an infinitely small part of the whole. Thus the circle is a subequal of the inscribed regular polygon with an infinite number of sides. No! surely the polygon is *sub*equal of the circle? Not so: the Latin preposition *sub*, thus used, does for both sides. *Subcruda* is used to mean short of raw on the cookery side, very little cooked; there is no under-raw meat: *coquito paulisper uti subcruda fiat.* Thus the area of a curve and the sum of the inscribed rectangles ydx are subequal each of the other.

obtained from all the other terms. And P is $1.2.3\ldots 2k \div (1.2)^k$ or $1.3.5\ldots 2k-1 \times 1.2.3\ldots k$. Hence $\Sigma(x+y+\ldots)^{2k}$ is subequal of $1\ldots 2k-1 \times 1\ldots k \times$ the sum of all products of $x^2, y^2, z^2, \ldots$ k and k together. And in this last we obtain $\Sigma{:}(x+y+\ldots)^{2k}$ by writing $\Sigma{:}x^2$ for x^2, &c. But by the preceding theorem $1\ldots k$ $\times$ (the sum of all products of $\Sigma{:}x^2$, &c. k and k together) is subequal of $(\Sigma{:}x^2+\Sigma{:}y^2+\ldots)^k$, or (since $\Sigma{:}xy=0$) of $\{\Sigma{:}(x+y+\ldots)^2\}^k$. Hence, if A_{2k} represent the average $2k$th power of all the values of $x+y+\ldots$ we have, if each letter be of balanced drawings,

$$A_{2k}=1.3.5\ldots 2k-1 A_2^k \quad . \quad . \quad . \quad . \quad (A)$$

In this and the preceding case the letters need not have the same set of values. Nevertheless, for a rough attempt at verification, let us suppose four letters, with drawings -1, 0, $+1$ for each. Then $x+y+z+v$ has 81 drawings; and if we write down each drawing as often as it occurs (counting negatives as positives, since we only want even powers) we have for the sum of the $2k$th powers of $x+y+z$

$$2.4^{2k}+8.3^{2k}+20.2^{2k}+32.1^{2k}+19.0^{2k}$$

giving 216, 1512, 15336, all divided by 81, or $\frac{8}{3}$, $\frac{56}{3}$, $\frac{568}{3}$, for attempts at A_2 A_4 A_6. Now $1.3A_2^2$ is $\frac{64}{3}$, not far from $\frac{56}{3}$; $1.3.5A_2^3$ is $\frac{7680}{27}$, instead of $\frac{5112}{27}$.

We now ask, what function of k, say ϕk, satisfies the functional equation (A). Without entering on the mode of finding, it will be enough here to state and verify that one solution is

$$A_{2k}=\sqrt{\frac{c}{\pi}}\int_{-\infty}^{\infty}\varepsilon^{-ct^2}t^{2k}dt.$$

It is a very common question to show, by integration by parts, that this gives

$$A_{2k}=\frac{2k-1}{2c}A_{2k-2}=\frac{2k-1}{2c}\cdot\frac{2k-3}{2c}A_{2k-4}=\ldots=\frac{2k-1}{2c}\cdots\frac{1}{2c}\cdot A_0.$$

And A_0 is known to be 1, whence A_2 is $\frac{1}{2c}$. Hence it appears that this form of A_{2k} gives $A_{2k}=(2k-1)\ldots 3.1.A_2^k$. The next question is, Are there any other solutions? Let the preceding be $\phi(2k)$, and let the most general value of A_{2k}, divided by $\phi(2k)$,

give $\psi(2k)$. We have then $A_{2k}=\phi(2k).\psi(2k)$; substitute this in (A) and strike out the equal factors $\phi 2k$ and $(2k-1)\ldots 1(\phi 2)^k$, which gives $\psi(2k)=(\psi 2)^k$, of which the only solution, so far as integers are concerned, is $\psi(2k)=m^{2k}$, m being any constant. The complete solution of (A) is then

$$A_{2k}=\sqrt{\frac{c}{\pi}}\cdot m^{2k}\int_{-\infty}^{+\infty}\varepsilon^{-ct^2}t^{2k}dt$$

where c and m are any constants.

We now come to another question. The number of letters which contribute their drawings to a sum being infinite (practically very many; or by the ascertained, but undemonstrated, goodness of the approximation, even a very moderate number) we have formed a law by which, as soon as the average *square* of the sum is known—which we must know to determine c—we determine the average $2k$th power of the sum. Our new question must be, Does this mode of proceeding implicitly contain a case of the problem? Have we found that, be the laws of the drawings what they may, the final law of the average $2k$th powers is the same as if those laws had all given way to some assignable law, deducible from the preceding formula. And the answer is affirmative, which may be shown as follows.

The factor m^{2k} may be thrown away. It merely indicates that if all the drawings be affected by a new factor, the average $2k$th power of the sum is affected by the $2k$th power of that factor. Now remember that if the drawings of a letter, a_1, a_2, &c., severally occur l_1, l_2, &c. times, the average $2k$th power of the drawings is $\Sigma la^{2k}:\Sigma l$, and* $l_w:\Sigma l$ is the multiplier for a_w^{2k} in that average power; and the sum of all the terms of the form $l_w:\Sigma l$ is unity. If then we find the factors λ_1, λ_2, &c. in $\lambda_1 a_1^{2k}+\ldots$ such that $\Sigma\lambda=1$, we have a representation of the average $2k$th power of the drawings, upon the supposition that a_1, a_2, &c. occur as possible drawings in numbers of times proportional to λ_1, λ_2, &c.; or, N being the whole number of drawings, $N\lambda_1$, $N\lambda_2$, &c. times.

It is well known that $\sqrt{\frac{c}{\pi}}\int_{-\infty}^{+\infty}\varepsilon^{-ct^2}dt=1$. Denote $\sqrt{\frac{c}{\pi}}\,\varepsilon^{-ct^2}$ by ft: then $\int_{-\infty}^{+\infty}ft.t^{2k}dt$ represents the average $2k$th power of a draw-

* The student who has had some glimpse of the problems in probability will wonder what I am at: I tell him I am keeping the theory of probability out of the way. Or rather;—When a barrister attempts to address the court without gown and wig, the judge says, Mr. ——, I can't *see* you! Suppose the theory of probability out of costume, and not privileged to be seen or heard. Perhaps I may make the application in another paper.

ing: the supposition being that every value of t occurs as a drawing $Nftdt$ times, where N, infinitely great, is the whole number of drawings. I cannot enter fully on this point: it must be enough here to say that the whole interval from $-\infty$ to $+\infty$ is divided into intervals of dt, as in $\ldots -3dt, -2dt, -dt, 0, dt, 2dt, 3dt, \ldots$ and the drawing zdt occurs times enough to make its collection the fraction $f(zdt)dt$ of the whole number of drawings.

We now see that the average $2k$th power of $(x+y+\ldots)$ is that of a drawing of one letter on the condition that every drawing is possible, and that the (infinite) number of drawings between a and b is the fraction $\sqrt{\frac{c}{\pi}}\int_a^b \varepsilon^{-ct^2}dt$ or $\int_a^b ft.dt$, of the infinite total number of drawings. And all we know of c is that $1:2c$ is the average square of a drawing, or $\int_{-\infty}^{+\infty} ft.t^2dt$.

If we take each one of the letters, $x, y, z, \ldots$ and take the average squares of their drawings separately, say $\frac{1}{2a}, \frac{1}{2\beta}$, &c., and then make $\frac{1}{2a}+\frac{1}{2\beta}+\ldots=\frac{1}{2c}$, we have the following result. Let λ be one of the letters $a, \beta, \ldots$, and for the law of drawings of x substitute that the fraction of drawings between a and b, for any values of a and b, is $\int_a^b f(t,\lambda)dt$; where $f(t,\lambda)$ is $\sqrt{\frac{\lambda}{\pi}}\,\varepsilon^{-\lambda t^2}$. Do the same with all the letters. Then, the sum has for its law of drawing that the fraction of its drawings which lies between a and b, is $\int_a^b f(t,c)dt$. And the average $2k$th powers, for an infinite number of drawings, are accurately represented—and for a moderate number very nearly—by the several cases of $\int_{-\infty}^{+\infty} f(t,\lambda)t^{2k}dt$. And $\int_{-\infty}^{+\infty} f(t,c)t^{2k}dt$ represents the average $2k$th power of the sum.

One case of the theory of probabilities is the application of the preceding to *balanced* errors of observation, in which positive and negative errors are equally likely. It is shown that everything depends upon average even powers of errors: whence from the preceding it is made to follow that, be the law of error what it may, the results, on a moderate number of observations, are the same as if it were that the chance of an error between a and b is $\sqrt{\frac{c}{\pi}}\int_a^b \varepsilon^{-ct^2}dt$, where $1:2c$ is the average square of an error.

There are various grounds on which this law of error lies under suspicion of being very near to a physical truth, by the nature of men and things: but the last theorem renders this of small importance. The proof of this last theorem, as given by Laplace, demands the highest resources of the integral calculus. Such a megatherium as

$$\frac{1}{\pi}\int_0^\pi \cos l\varpi\, d\varpi \left[\psi\left(\frac{0}{n}\right)+\psi\left(\frac{1}{n}\right)\cdot 2\cos\varpi+\ldots\psi\left(\frac{n}{n}\right)\cdot 2\cos n\varpi\right]^s$$

must not be exhibited in an elementary museum.

A few years ago (*Camb. Trans.*, vol. x., part 2, Nov. 11, 1861) I succeeded in reducing the connexion of all laws with the *final law* to the comparatively easy form shown in this paper. The considerations introduced are such as may become useful in actuaries' work: and certainly will, if the higher mathematics continue to be applied in life contingencies as they have been of late years.

Difficulty may present itself to those who have no sufficient command of an integral as the sum of an infinitely great number of infinitely small elements. I shall not, for instance, be quite clear throughout this paper to a person on whom it does not flash, when I state it, that I have proved the following theorem:—If $\int_a^b \phi x dx$ be distributed into an infinite number of elements of the form $\phi x dx$, and if every combination of k elements be multiplied together, the sum of all the products is $\left(\int_a^b \phi x dx\right)^k \div 1.2\ldots k$. Required a very simple proof of this.

An instance of verification is always valuable; and I therefore give one case of each leading theorem, using the abbreviations of the *calculus of operations*.

Let each of the σ letters $x, y, \ldots$ give the drawings 0 and 1. The sum of the kth powers of all drawings of $x+y+\ldots$ is $0^k+1_\sigma 1^k+2_\sigma 2^k+\ldots+\sigma_\sigma \sigma^k$. Let E represent the operation of changing m into $m+1$: the above is the operation $(1+\mathrm{E})^\sigma$ performed upon 0 in 0^k. This is $(2+\Delta)^\sigma.0^k$; and since $\Delta^n 0^k=0$ when $n>k$, the highest term is $k_\sigma 2^{\sigma-k}\Delta^k 0^k$, which, since σ is infinite and $\Delta^k 0^k=1.2\ldots k$, is $\sigma^k 2^{\sigma-k}$. Divide by 2^σ, the number of drawings of $x+y+\ldots$, and we have $(\frac{1}{2}+\frac{1}{2}+\ldots\ \sigma \text{ terms})^k$, or the kth power of the sum of the averages.

Again, let each of the σ letters have the drawings -1 and $+1$. The sum of the $2k$th powers of the drawings of $x+y+\ldots$ is $\sigma^{2k}+1_\sigma(\sigma-2)^{2k}+2_\sigma(\sigma-4)^{2k}+\ldots+\sigma_\sigma(-\sigma)^{2k}$ which is $(\mathrm{E}+\mathrm{E}^{-1})^\sigma$

applied to 0 in 0^{2k}. This is $\{2+\Delta^2(1+\Delta)^{-1}\}^{\sigma}0^{2k}$; and its highest term has $\Delta^{2k}(1+\Delta)^{-k}0^{2k}$, in the development of which only $\Delta^{2k}0^{2k}$ has value. The term to be retained is therefore $k_{\sigma}2^{\sigma-k}\Delta^{2k}0^{2k}$, or, σ being infinite,

$$\sigma^k 2^{\sigma-k}\Delta^{2k}0^{2k} \div 1.2\ldots k, \text{ or } \sigma^k 2^{\sigma-k} \times 1.3.5\ldots 2k-1 \times 2^k,$$

$\Delta^{2k}0^{2k}$ being $1.2.3\ldots 2k$. Divide by 2^{σ}, the number of drawings of $x+y+\ldots$ and we have $\sigma^k \times 1.3\ldots 2k-1$, in which σ, or $(1+1+1+\ldots\ \sigma$ terms), is the sum of the average squares of the drawings of $x, y, \ldots$

The theorem which contains the signification of $\sigma(\sigma-1)\ldots(\sigma-a-b-c-\ldots+1)$ divided by $[a.[b.[c.\ldots$ seldom or never finds a place in chapters on combination. I recommend it to the attention of the elementary student, who may find various demonstrations of it.

On the Rate of Interest in Loans repayable by Instalments. By PETER GRAY, F.R.A.S., *Honorary Member of the Institute of Actuaries.*

(Continued from page 102.)

MY second example is the Austrian Loan of 1865. The conditions are as follows:—

734,694 Bonds, each of £19. 17*s*. 0*d*., issued at £13. 14*s*. 4*d*., that is, at a discount of £6. 2*s*. 8*d*. each.

£	s.	d.	
£1	0	0	per Bond to be paid on application, say Dec. 1, 1865, and the remaining £12. 14*s*. 4*d*. in the following instalments:—
1	19	7	on Dec. 15, 1865,
3	11	7	„ Feb. 10, 1866,
3	11	7	„ April 10, „ ,
3	11	7	„ June 10, „ ,
13	14	4	Total.

Subscribers will be at liberty to pay their Scrip in full on any one of the above dates, under discount at 6 per cent per annum.

The bonds are to bear interest at the rate of 9*s*. 11*d*. each (=·4958333) per half year, (a trifle under $2\frac{1}{2}$ per cent on the *nominal* amount) payment of which to become due on the 1st June and the 1st Dec. of each year.

The bonds are to be redeemed in thirty-seven years, 9928 (to be selected by lot) half-yearly, at the same dates as the payments of

interest. The first drawing to take place in May, 1868, and the bonds then drawn to be paid off on the 1st of June thereafter.

Required the cost per cent of the loan to the borrower, and the rates realized on the bonds paid off each half year.

Here it will be observed that the number of the bonds being 734,694, while $9928 \times 74 = 734,672$, the payment of 22 bonds is left unprovided for. I shall assume that these 22 bonds are to be included in the last payment, making the number paid off at the end of the 78th half year, 9950.

The principal points in which this problem differs from the former are that here, first, the loan is issued at a discount; secondly, the repayments of principal do not commence immediately; and thirdly, that these repayments, (with the exception of the last—see above,) are uniform.

To determine the cost to the borrower we have to find the rate at which the values, at any epoch, of the sums receivable and payable by him are equal to each other. The most convenient epoch of reference, in this case as in the last, is the commencement of the transaction—the date of issue, Dec. 1st, 1865.

The receipts are reduced to this epoch as follows :—

Dec. 1 . .	734,694·000		
15 . .	1,454,081·875	14 days	3,346·380
Feb. 10 . .	2,629,592·275	71 „	30,690·584
Apr. 10 . .	2,629,592·275	130 „	56,194·027
June 10 . .	2,629,592·275	191 „	82,561·993
	10,077,552·700		172,792·984
	172,792·984		
	9,904,759·716		

The reduction is effected by discounting at 6 per cent, (the stipulated rate) the several instalments for the time to elapse between the epoch of reference and their respective dates of payment. The discount amounts to 172,793, deducting which from (£13. 14*s.* 4*d.* $\times$ 734,694 $=$) 10,077,553, the difference, 9,904,760, is the value at the chosen epoch of the borrower's receipts.

Next, to find the value of his payments at the same epoch. It will be convenient to consider these in three portions. The first is a uniform annuity, for four half-yearly terms, of the interest on the bonds; the second is a variable annuity deferred four terms, and to last for seventy-four, whose payments are, for each term, the sum of the interest on the principal unpaid at the beginning of that term and the amount of the bonds repayable at the end of

it; and the third is the amount of the 22 residual bonds repayable at the end of the seventy-eighth term. I will deal with these in order.

First, the uniform annuity. This consists of four half-yearly payments of ·495833 × 734,694 = 364,285·775; and if we denote the required half-yearly rate by i, its value will be,

$$364{,}285{\cdot}775\,\frac{1-v^4}{i}.$$

Secondly, the variable annuity, to be entered upon in two years, and to make 74 half-yearly payments. I shall here use the theorem (1), which, adapted to this case, is,

$$v^4\left\{\frac{b_1}{i}+\frac{\Delta b_1}{i^2}+\ldots-v^{74}\left(\frac{b_{75}}{i}+\frac{\Delta b_{75}}{i^2}+\ldots\right)\right\}.$$

We shall determine b_1, &c., as follows:—

Int. on	734,694	bonds	@ ·495833	364,285·775	
Payable	9,928	„	@ 19·85	197,070·800	
				561,356·575	$=b_1$
Int. on	724,766	„	@ ·495833	359,363·142	
Payable	9,928	„	19·85	197,070·800	
				556,433·942	$=b_2$

It is unnecessary to go further. We see that b_2-b_1, or $\Delta b_1=-4922{\cdot}633$, which is the half-yearly interest on the 9928 bonds paid off the preceding year; and each succeeding payment will evidently have the same difference. Hence $\Delta^2 b_1$, &c. $=0$.

Hence also,

$$\begin{aligned} b_{75}&=b_1+74\Delta b_1,\\ &=561{,}356{\cdot}575-4922{\cdot}633\times 74,\\ &=561{,}356{\cdot}575-364{,}274{\cdot}867,\\ &=197{,}081{\cdot}708,\\ \text{and }\ \Delta b_{75}&=-4922{\cdot}633. \end{aligned}$$

Hence the value of this annuity is

$$v^4\left\{\frac{561{,}356{\cdot}6}{i}-\frac{4922{\cdot}633}{i^2}-v^{74}\left(\frac{197{,}081{\cdot}7}{i}-\frac{4922{\cdot}633}{i^2}\right)\right\}.$$

Thirdly, the value of the 22 bonds due at the end of the 78th half year is,

$$22\times 19{\cdot}85\times v^{78}=436{\cdot}70v^{78}.$$

The sum of the three results, namely,

$$\frac{364,285\cdot8}{i}+\frac{197,070\cdot8v^4}{i}-\frac{4922\cdot633v^4}{i^2}-\frac{197,081\cdot7v^{78}}{i}+\frac{4922\cdot633v^{78}}{i^2}+436\cdot7v^{78},$$

or,

$$\frac{364,285\cdot8}{i}+\frac{197,070\cdot8}{i(1+i)^4}-\frac{4922\cdot633}{i^2(1+i)^4}-\frac{197,081\cdot7}{i(1+i)^{78}}+\frac{4922\cdot633}{i^2(1+i)^{78}}+\frac{436\cdot7}{(1+i)^{78}},$$

is the value of the borrower's payments at the epoch of reference, in terms of i. And we have now to find the value of i which makes this expression equal to 9,904,760. This is done by trial, and the operation is as follows:—

i	·04	·045	·0437	·0436473	·0436484
log i	$\bar{2}$·6020600	$\bar{2}$·6532125	$\bar{2}$·6404814	$\bar{2}$·6399574	$\bar{2}$·6399683
" $(1+i)$	0·0170333	0·0191163	0·0185757	0·0185538	0·0185542
" $(1+i)^4$	0·0681334	0·0764652	0·0743027	0·0742150	0·0742168
" $i(1+i)^4$	$\bar{2}$·6701934	$\bar{2}$·7296777	$\bar{2}$·7147841	$\bar{2}$·7141724	$\bar{2}$·7141851
" $i^2(1+i)^4$	$\bar{3}$·2722534	$\bar{3}$·3828902	$\bar{3}$·3552655	$\bar{3}$·3541298	$\bar{3}$·3541534
" $(1+i)^{78}$	1·3286005	1·4910706	1·4489030	1·4471933	1·4472285
" $i(1+i)^{78}$	$\bar{1}$·9306605	0·1442831	0·0893844	0·0871507	0·0871968
" $i^2(1+i)^{78}$	$\bar{2}$·5327205	$\bar{2}$·7974956	$\bar{2}$·7298658	$\bar{2}$·7271081	$\bar{2}$·7271651

364285·8	5·5614423	5·5614423	5·5614423	5·5614423	5·5614423
	$\bar{2}$·6020600	$\bar{2}$·6532125	$\bar{2}$·6404814	$\bar{2}$·6399574	$\bar{2}$·6399683
A	6·9593823	6·9082298	6·9209609	6·9214849	6·9214740
197070·8	5·2946223	5·2946223	5·2946223	5·2946223	5·2946223
	$\bar{2}$·6701934	$\bar{2}$·7296777	$\bar{2}$·7147841	$\bar{2}$·7141724	$\bar{2}$·7141851
B	6·6244289	6·5649446	6·5798382	6·5804499	6·5804372
4922·633	3·6921974	3·6921974	3·6921974	3·6921974	3·6921974
	$\bar{3}$·2722534	$\bar{3}$·3828902	$\bar{3}$·3552655	$\bar{3}$·3541298	$\bar{3}$·3541534
C	6·4198440	6·3093072	6·3369319	6·3380676	6·3380440
197081·7	5·2946463	5·2946463	5·2946463	5·2946463	5·2946463
	$\bar{1}$·9306605	0·1442831	0·0893844	0·0871507	0·0871968
D	5·3639858	5·1503632	5·2052619	5·2074956	5·2074495
4922·633	3·6921974	3·6921974	3·6921974	3·6921974	3·6921974
	$\bar{2}$·5327205	$\bar{2}$·7974956	$\bar{2}$·7298658	$\bar{2}$·7271081	$\bar{2}$·7271651
E	5·1594769	4·8947018	4·9623316	4·9650893	4·9650323
436·7	2·6401832	2·6401832	2·6401832	2·6401832	2·6401832
	1·3286005	1·4910706	1·4489030	1·4471933	1·4472285
F	1·3115827	1·1491126	1·1912802	1·1929899	1·1929547

A	9107100	8095242	8336062	8346126	8345916
B	4211400	3672353	3800477	3805833	3805723
C	2629900	2038484	2172360	2178040	2177930
D	231200	141372	160421	161248	161231
E	144400	78470	91692	92276	92264
F	20	14	16	16	16
	13462920	11846079	12228247	12244251	12243919
	2861100	2179856	2332781	2339288	2339161
Result	10601820	9666223	9895466	9004963	9904758
	9904760	9904760	9904760	9904760	9904760
Error	697060	238537	9294	203	2

I commence my trials with $i = \cdot 04$, conjecturably near the truth. It turns out to be too small, giving an error of 697060. I then try ·045, which is too great, error, 238537. I then use *False Position*, which, as applied to problems such as the present gives the following rule:—Multiply the error of the last result by the last correction of the rate, and divide the product by the difference of the last two results. The quotient will be a new correction to be applied in either augmentation or diminution of the rate last used, according as that rate was found to be too small or too great.

We thus have here

$$\frac{238537 \times \cdot 005}{10601820 - 9666223} = \frac{1192 \cdot 685}{935597} = \cdot 0013.^*$$

And $\cdot 045 - \cdot 0013 = \cdot 0437$ is a corrected rate. This is found on trial to be still too great, but the error is reduced to 9294. For a new correction we have

$$\frac{9 \cdot 294 \times \cdot 0013}{9895 - 9666} = \frac{\cdot 01208}{229} = \cdot 0000527.$$

Hence $\cdot 0437 - \cdot 0000527 = \cdot 0436473$ is a second corrected rate, trial of which shows that it is a trifle too small, the error being 203.

It would hardly be considered necessary to seek a closer approximation than this. I have nevertheless, as a matter of curiosity, carried the operation a step further. Thus:—

$$\frac{203 \times \cdot 0000527}{9904963 - 9895466} = \frac{\cdot 01070}{9497} = \cdot 0000011.$$

* As the quotients in this operation are taken to only two or three places, it is unnecessary to use more than three or four figures in the numbers producing them. Thus, instead of the above we might have written

$$\frac{238 \cdot 5 \times \cdot 005}{10602 - 9666} = \frac{1 \cdot 193}{936} = \cdot 0013.$$

This gives $\cdot 0436473 + \cdot 0000011 = \cdot 0436484$ for a third corrected rate, and trial of this rate gives, as shown, an error of 2.* It thus appears that the method employed enables us, by three or four trials, to assign the required rate to any degree of approximation that may be desired.

It is the half-yearly rate that has just been determined. The yearly rate is easily deduced from it thus:—

$$(1\cdot 0436484)^2 - 1 = \cdot 0892018, \text{ the yearly rate;}$$

that is, 8·92018 per cent.

The foregoing loan was brought out in the latter end of 1865, and it gave rise to some correspondence in the *Times*. In the paper of Nov. 30 "A Broker," (under date Nov. 28,) gives his calculation of the rate somewhat as follows:—

Bonds 734,694 at £13. 14*s*. 4*d*. = £10,077,552. Annual interest £723,571 = 7·23 per cent per annum. Bonds to be paid off at £19. 17*s*. 0*d*. each, (an increase of 45 per cent,) in thirty-seven years from 1st June, 1868, in equal half-yearly payments. The amount paid for interest and the bonus on the principal will thus be the same as though the entire loan were retained for 18½ years from that date and then paid off.

Loan completed June, '66; interest to Dec. 1, '86, 20½ years at 9*s*. 11*d*. each bond per half year will be		14,935,716
Nominal Principal paid, 734,694 bonds, at £19. 17*s*. 0*d*. each . .	14,583,675	
Amount really borrowed . .	10,077,552	
		4,506,123
Amount paid for Interest and Bonus . .		19,441,839,

or at the rate of 9·40 per cent on the real amount of the loan.

The above is presented, I suppose, as only an approximation. I cannot say that I follow the author's reasoning; but it must be admitted that his result is, all things considered, wonderfully near the truth.

Dissatisfied apparently with "A Broker's" result, "An Actuary" takes up the matter in the *Times* of Dec. 4. I have not the details of his calculation, but the following is his conclusion:—"I think it will be seen from the foregoing that the cost to the borrower is, in round numbers, 7½ per cent per annum." The only observation I make on this is, that it affords illustration of a remark made towards the commencement of this paper, that the approximate

* In making the trial it is in this case necessary to form $\log(1+i)$ to 9 places in order to have $\log(1+i)^{78}$ true in the seventh place.

methods used in practice for the solution of such problems as the present give results entitled to but little confidence.

It remains to determine the rates that will be realized on the several bonds, which rates are dependent upon the periods at which the bonds are respectively paid off. The value of the investment in respect of each bond on the 1st December, 1865, is (9,904,760 ÷ 734,694 =) 13·48148, and if paid off at the end of the nth half year the investor receives a half-yearly payment of (9*s.* 11*d.* =) ·495833 during the term, and at the end of it 19·85. Calling, as before, i the required half-yearly rate, we have for the values of the repayments at the same epoch,

$$\frac{\cdot 495833(1-v^n)}{i}, \text{ and } 19\cdot 85v^n,$$

respectively. Hence the equation for determining i is

$$\frac{\cdot 495833(1-v^n)}{i} + 19\cdot 85v^n = 13\cdot 48148$$

or,
$$\frac{\cdot 1-v^n}{i} + 40\cdot 03362v^n = 27\cdot 18954.$$

From this the value of i corresponding to each admissible value of n must, as in previous cases, be found by trial. To avoid unnecessary labour, and as a guide in our trials, find first the limits within which the values of i are contained. For $n=1$ the foregoing becomes

$$41\cdot 0336v = 27\cdot 1895$$

or
$$27\cdot 1895(1+i) = 41\cdot 0336;$$

from which we easily find $i = \cdot 50917$.

And if $n = \infty$ the expression becomes

$$\frac{1}{i} = 27\cdot 1894,$$

whence
$$i = \cdot 03678.$$

These are wide limits. But observing that from the nature of the case the values must decrease rapidly from the greater limit as n increases, that is as the period of repayment of the principal becomes more remote, it seems probable that for $n=5$, (the first value with which we have to do,) 10 per cent, that is, $i = \cdot 1$, will be pretty near the truth. Trying this we get for result 28·6485. This value of i is too small. Trying ·11 we get 27·4533, and we are still under the mark. The error being 27·4533 − 27·1895 = ·2638, we find a correction thus :—

$$\frac{\cdot 2638 \times \cdot 1}{28\cdot 6485 - 27\cdot 4533} = \cdot 00221,$$

which gives for a new value of i, $\cdot 11 + \cdot 00221 = \cdot 11221$. Trying this we get for result $27 \cdot 1983$, which is quite near enough, the error being only $= \cdot 0088$. I have computed a few cases, and consigned the results to the small table below; but it is unnecessary to give further details. The annual rate is found from the half-yearly rate by the formula, $(1+i)^2 - 1$.

n.	Half-yearly Rate per Cent.	Error.	Annual Rate per Cent.
5	11·221	+·0088	23·700
25	4·708	+·0008	9·635
45	4·062	−·0022	8·288
65	3·849	−·0019	7·845
78	3·783	−·0055	7·709
Limit	3·678		7·490

On the Theory of Annuities Certain. By WILLIAM MATTHEW MAKEHAM, *Fellow of the Institute of Actuaries.*

THE interest with which I have perused Mr. Gray's very able paper in the last number of the *Journal,* has led me to give some consideration to the general theory of varying Annuities; and as the results of the investigation appear to me somewhat striking, I am induced to think that a brief paper on the subject may not be unacceptable to that section of the scientific world to which this *Journal* more particularly addresses itself.

Nearly all that has hitherto appeared in connection with this subject is comprised, I believe, in Chapters XV. and XVI. of Baily's work on Interest and Annuities.* Like everything which proceeded from the hand of that excellent writer, the matter is treated with skill and perspicuity; but owing to the prevailing want of appreciation of the importance of the inquiry (an importance which Mr. Gray has been the first to demonstrate), and the restricted scope of the object which the author had in consequence proposed to himself, the result is very far indeed from satisfactory. In proof of this assertion I need only mention the fact that

* This remark is applicable only to Annuities Certain, as I find that Mr. Gray himself, in his work on Life Contingencies, has treated upon Life Annuities the successive payments of which are the several orders of figurate numbers. The connection between the values of Annuities of two successive orders is shewn by Mr. Gray to be expressed by the equation $\dot{i}_x^{(m)} = \dot{i}_x^{(m-1)} + vp_x \dot{i}_{x+1}^{(m)}$, where $\dot{i}_x^{(m)}$ denotes the value of an annuity of the mth order of figurate numbers, on a life aged x. In Mr. G.'s arrangement the payment due at the end of the first year is always unity.

Mr. Gray, in investigating his problem, has not attempted to avail himself of Baily's researches, but has applied a very general and useful formula which he had himself brought forward in a former paper devoted to a totally different subject. Having a specific object in view, viz., the numerical solution of a particular problem which had been erroneously treated by another writer, Mr. Gray has naturally used his formula in its readiest form, and has not gone out of his way to trace its connection with the general system by which the doctrine of Annuities is usually expounded. If therefore it shall appear that by following a different course a more convenient expression might have been obtained, such a circumstance in no wise detracts from the merits of Mr. Gray's performance, seeing that the labour of the necessary preliminary investigation would have greatly outweighed any saving which might thereby have been effected in the numerical solution of the problem.

Lemma. (I.)

Let an annuity the xth payment of which is unity, be termed an annuity of the first order; an annuity, the xth payment of which is $(x-1)$, an annuity of the second order; and, generally, an annuity the xth payment of which is $\frac{(x-1)(x-2)\ldots(x-t+1)}{1.2\ldots.(t-1)}$, an annuity of the tth order. Also let $(1+i)^n$ be represented by $\overset{0}{A}_n$, and let $\overset{1}{A}_n$ denote the amount of an annuity (for n years) of the first order, $\overset{2}{A}_n$ the amount of an annuity (for the like term) of the second order, and so on. Further let $\overset{t}{[x]}$ denote the xth payment of an annuity of the tth order. Then I say that the law of the series $\overset{0}{A}_n$, $\overset{1}{A}_n$, $\overset{2}{A}_n$ .... shall be expressed by the equation

$$\overset{t}{A}_n = \frac{\overset{t-1}{A}_n - \overset{t}{[n+1]}}{i}$$

Demonstration.

When $t=1$, the equation becomes $\overset{1}{A}_n = \frac{\overset{0}{A}_n - \overset{1}{[n+1]}}{i} = \frac{(1+i)^n - 1}{i}$, the truth of which is proved in all works on the doctrine of Annuities Certain. The following Table shewing the successive payments of Annuities of different orders will be useful in dealing with higher values of t.

x.	1st Order.	2nd Order.	3rd Order.	4th Order.	5th Order.	6th Order.	7th Order.	8th Order.
1	1	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0
3	1	2	1	0	0	0	0	0
4	1	3	3	1	0	0	0	0
5	1	4	6	4	1	0	0	0
6	1	5	10	10	5	1	0	0
7	1	6	15	20	15	6	1	0
8	1	7	21	35	35	21	7	1
&c.	1	&c.	&c.	&c.	&c.	&c.	&c.	&c.

By a well known property of these series we have

$$\overset{t}{[n]}=\overset{t}{[n-1]}+\overset{t-1}{[n-1]}.$$

Now, $\overset{t}{A}_n=\overset{t}{[n]}+\overset{t}{[n-1]}(1+i)+\overset{t}{[n-2]}(1+i)^2+\ldots+\overset{t}{[1]}(1+i)^n.$

And $\overset{t-1}{A}_n=\overset{t-1}{[n]}+\overset{t-1}{[n-1]}(1+i)+\overset{t-1}{[n-2]}(1+i)^2+\ldots+\overset{t-1}{[1]}(1+i)^n.$

Hence, by addition,

$$\overset{t}{A}_n+\overset{t-1}{A}_n=\overset{t}{[n+1]}+\overset{t}{[n]}(1+i)+\overset{t}{[n-1]}(1+i)^2+\ldots+\overset{t}{[2]}(1+i)^n=\overset{t}{A}_{n+1},$$

as for all values of t greater than 1, $\overset{t}{[1]}$ vanishes. Therefore

$$\overset{t}{A}_n+\overset{t-1}{A}_n=\overset{t}{A}_{n+1}.$$

But $$\overset{t}{A}_{n+1}=\overset{t}{A}_n(1+i)+\overset{t}{[n+1]}.$$

Therefore $$\overset{t-1}{A}_n=\overset{t}{A}_n i+\overset{t}{[n+1]};$$

and finally, $$\overset{t}{A}_n=\frac{\overset{t-1}{A}_n-\overset{t}{[n+1]}}{i}$$

Cor. Putting t successively equal to 1, 2, 3, 4, &c., we have

$$\overset{1}{A}_n=\frac{(1+i)^n-1}{i}$$

$$\overset{2}{A}_n=\frac{\overset{1}{A}_n-n}{i}$$

$$\overset{3}{A}_n=\frac{\overset{2}{A}_n-n\dfrac{n-1}{2}}{i}$$

$$\overset{4}{A}_n=\frac{\overset{3}{A}_n-n\dfrac{n-1}{2}\cdot\dfrac{n-2}{3}}{i}$$

&c.= &c.

As an example of the use of this theorem let it be required to find the successive values of $\overset{2}{A}_5, \overset{3}{A}_5 \ldots \overset{5}{A}_5$, when $i = \cdot 05$.

$n=5$ $\qquad \overset{1}{A}_5 = 5\cdot 52563125$ (Baily, *Int. & Anns.*, Table V.)

$\qquad\qquad \cdot 52563125 \times 20$

$n\frac{n-1}{2} = 10 \qquad \overset{2}{A}_5 = 10\cdot 5126250$

$\qquad\qquad \cdot 5126250 \times 20$

$n\frac{n-1}{2}\cdot\frac{n-2}{3} = 10 \qquad \overset{3}{A}_5 = 10\cdot 252500$

$\qquad\qquad \cdot 252500 \times 20$

$n\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4} = 5 \qquad \overset{4}{A}_5 = 5\cdot 05000$

$\qquad\qquad \cdot 05000 \times 20$

$\qquad \overset{5}{A}_5 = 1\cdot 0000$

The final result proves the accuracy of the process, as $\overset{n}{A}_n$ is always unity.

LEMMA. (II.)

Let unity be denoted by $\overset{0}{V}_n$ and let $\overset{1}{V}_n$ denote the value of an Annuity of the first order, $\overset{2}{V}_n$ the value of an annuity of the second order, and so on. The law of the series $\overset{0}{V}_n, \overset{1}{V}_n, \overset{2}{V}_n \ldots\ldots$ shall be expressed by the equation:

$$\overset{t}{V}_n = \frac{\overset{t-1}{V}_n - [\overset{t}{n}+1](1+i)^{-n}}{i}$$

Demonstration.

$$\overset{t}{V}_n \quad = [\overset{t}{1}](1+i)^{-1} + [\overset{t}{2}](1+i)^{-2} + \ldots + [\overset{t}{n}](1+i)^{-n}$$

$$\overset{t-1}{V}_n \quad = [\overset{t-1}{1}](1+i)^{-1} + [\overset{t-1}{2}](1+i)^{-2} + \ldots + [\overset{t-1}{n}](1+i)^{-n}$$

$$\therefore \overset{t}{V}_n + \overset{t-1}{V}_n = [\overset{t}{2}](1+i)^{-1} + [\overset{t}{3}](1+i)^{-2} + \ldots + [\overset{t}{n}+1](1+i)^{-n} = \overset{t}{V}_{n+1}(1+i).$$

But $\overset{t}{V}_{n+1} = \overset{t}{V}_n + [\overset{t}{n}+1](1+i)^{-\overline{n+1}}$. Hence $\overset{t}{V}_n + \overset{t-1}{V}_n = \overset{t}{V}_n(1+i) + [\overset{t}{n}+1](1+i)^{-n}$ and $\overset{t}{V}_n i = \overset{t-1}{V}_n - [\overset{t}{n}+1](1+i)^{-n}$. Finally we have

$$\overset{t}{V}_n = \frac{\overset{t-1}{V}_n - [\overset{t}{n}+1](1+i)^{-n}}{i}$$

The case in which $t=1$ is proved in works on Annuities Certain,—the equation then becoming $\overset{1}{V}_n=\dfrac{1-(1+i)^{-n}}{i}$.

Cor. Putting t successively equal to 1, 2, 3, 4, &c. we have

$$\overset{1}{V}_n=\frac{1-(1+i)^{-n}}{i}$$

$$\overset{2}{V}_n=\frac{\overset{1}{V}_n-n(1+i)^{-n}}{i}$$

$$\overset{3}{V}_n=\frac{\overset{2}{V}_n-n\dfrac{n-1}{2}(1+i)^{-n}}{i}$$

$$\overset{4}{V}_n=\frac{\overset{3}{V}_n-n\dfrac{n-1}{2}\cdot\dfrac{n-2}{3}(1+i)^{-n}}{i}$$

$$\&c.= \qquad \&c.$$

In illustration of this Theorem I propose to determine successively the values of $\overset{1}{V}_{40}$, $\overset{2}{V}_{40}$, and $\overset{3}{V}_{40}$ when interest is taken at 5 per cent.

$$\begin{array}{rr}
(1+i)^{-40}= & \cdot 14204568 \\ \hline
40(1+i)^{-40}= & 5\cdot 6818272 \\ \hline
 & 113\cdot 636544 \\
 & 2\cdot 840914 \\ \hline
40\times\dfrac{39}{2}(1+i)^{-40}= & 110\cdot 795630
\end{array}$$

$$\begin{array}{rl}
 & 1\cdot \\
 & \cdot 1420457 \\ \hline
 & \cdot 8579543\times 20 \\ \hline
\overset{1}{V}_{40}= & 17\cdot 159086 \\
 & 5\cdot 681827 \\ \hline
 & 11\cdot 477259\times 20 \\ \hline
\overset{2}{V}_{40}= & 229\cdot 54518 \\
 & 110\cdot 79563 \\ \hline
 & 118\cdot 74955\times 20 \\ \hline
\overset{3}{V}_{40}= & 2374\cdot 9910
\end{array}$$

Problem. (I.)

To determine the amount of an annuity for n years, the successive payments of which are $u_1, u_2, u_3 \ldots u_n$.

Solution.

Substituting for u_2, u_3, &c. their respective values in terms of u_1, Δu_1, $\Delta^2 u_1$, &c. we have

$$u_1 = u_1$$
$$u_2 = u_1 + \Delta u_1$$
$$u_3 = u_1 + 2\Delta u_1 + \Delta^2 u_1$$
$$u_4 = u_1 + 3\Delta u_1 + 3\Delta^2 u_1 + \Delta^3 u_1$$
$$\cdot \quad \cdot$$
$$u_n = u_1 + (n-1)\Delta u_1 + (n-1)\frac{n-2}{2}\Delta^2 u_1 + \ldots. + \Delta^{n-1} u_1$$

Hence it appears that the given annuity is equivalent to an annuity of the first order, for n years, of u_1, + an annuity of the second order, for the same term, of Δu_1, + an annuity of the third order, for the same term, of $\Delta^2 u_1$, and so on until the last significant order of differences. The required amount is therefore

$$\overset{1}{A}_n . u_1 + \overset{2}{A}_n . \Delta u_1 + \overset{3}{A}_n . \Delta^2 u_1 + \ldots.$$

Example.—Required the amount of an annuity for 5 years at 5 per cent the successive payments of which are 5^3, 4^3, 3^3, 2^3, 1^3.

Here we have

$$\begin{array}{lllll} u_1 = 125 & & & & \\ & -61 & & & \\ u_2 = 64 & & +24 & & \\ & -37 & & -6 & \\ u_3 = 27 & & +18 & & \\ & -19 & & -6 & \\ u_4 = 8 & & +12 & & \\ & -7 & & & \\ u_5 = 1 & & & & \end{array}$$

and therefore the amount of the given annuity is

$$125\overset{1}{A}_5 - 61\overset{2}{A}_5 + 24\overset{3}{A}_5 - 6\overset{4}{A}_5.$$

The example in the case of Theorem I. gives the values of $\overset{1}{A}_5$, &c.

5·52563125 × 125 — 10·512625 × 61 + 10·2525000 × 24 — 5·050000 × 6

1·38140781	630·757500	———	———
———	———	205·050000	30·300000
690·703906	641·270125	41·010000	
246·060000	30·300000	———	
———	———	246·060000	
936·763906	671·570125		
671·570125			
———			
265·193781 (Answer.)			

When there are three orders of differences and only five terms little is gained by the use of the formula, as each term of the annuity may easily be valued separately. I have however selected this example because it is one given by Baily, who gets for the answer 265·19378125,—showing that mine is correct to the last decimal place. Baily probably selected this case for the facility of

testing his result; a very proper precaution for an author to observe on breaking new ground.

I was surprised to observe that Baily designates the above an *Increasing Annuity*,—by which term (when treating of the *amounts* of annuities) he invariably understands an annuity whose last payment is unity and whose preceding payments are greater than unity.*

PROBLEM. (II.)

To determine the present value of an annuity (for n years) the successive payments of which are $u_1, u_2, u_3 \ldots u_n$.

Solution.

Substituting for u_2, u_3, &c. their respective values in terms of u_1, Δu_1, &c., as in the solution of the preceding Problem, and taking the present value of each order, we have for the value required

$$\overset{1}{\mathrm{V}}_n . u_1 + \overset{2}{\mathrm{V}}_n . \Delta u_1 + \overset{3}{\mathrm{V}}_n . \Delta^2 u_1 + \ldots .$$

Example.—Required the value of an annuity for 40 years at 5 per cent. the successive payments of which are 417500, 424680, 431620, &c. the third differences of the series being 0.

Differencing we have

$$\begin{array}{lll} u_1 = 417500 & & \\ & +7180 & \\ u_2 = 424680 & & -240 \\ & +6940 & \\ u_3 = 431620 & & \end{array}$$

Hence the required present value is $417500 \times \overset{1}{\mathrm{V}}_{40} + 7180 \times \overset{2}{\mathrm{V}}_{40} - 240 \times \overset{3}{\mathrm{V}}_{40}$; and taking the values of $\overset{1}{\mathrm{V}}_{40}$, &c., from the example given in the second Theorem we have

$$\begin{array}{lll} \log \overset{1}{\mathrm{V}}_{40} = 1{\cdot}2344943 & \log \overset{2}{\mathrm{V}}_{40} = 2{\cdot}3608683 & \log \overset{3}{\mathrm{V}}_{40} = 3{\cdot}3756620 \\ \log u_1 = 5{\cdot}6206565 & \log \Delta u_1 = 3{\cdot}8561244 & \log \Delta^2 u_1 = 2{\cdot}3802112 \\ \hline 6{\cdot}8551508 & 6{\cdot}2169927 & 5{\cdot}7558732 \end{array}$$

* Since writing the above Mr. Adler has shewn me a presentation copy of Baily's work, which belonged to the late Mr. Gompertz, in which the whole of the investigation relative to the *amounts* of increasing annuities is cancelled and the following note, in Baily's own handwriting, appended: "The whole of this investigation is erroneous "*in principle*,—it was written in haste whilst the proof was waiting. (Signed) F. B." This explains the circumstance alluded to in the text.

Mr. Adler has also pointed out to me that the materials of Baily's two chapters are to be found in the second volume of Dodson's *Mathematical Repository*, to which work, indeed, Baily refers his readers for further information on the subject.

$$\begin{array}{r} +7{,}163{,}922 \\ +1{,}648{,}135 \\ \hline 8{,}812{,}057 \\ -\;569{,}998 \\ \hline 8{,}242{,}059 \end{array} \quad \text{(Answer.)}$$

This is one of the values determined by Mr. Gray in the solution of his problem.

If it be required to determine the present value of a perpetuity the successive payments of which are $u_1\ u_2\ u_3 \ldots$ (a series of which the differences are supposed to vanish after a definite number of orders) we are led into a discussion which demands the application of the higher mathematics. In the case of Annuities of the first order, no difficulty arises; for $\overset{1}{\mathrm{V}}_n$ being $\frac{1-(1+i)^{-n}}{i}$, the effect of making n infinitely great is evident by inspection, as the term $(1+i)^{-n}$ vanishes, and we have $\overset{1}{\mathrm{V}}_\infty = \frac{1}{i}$, as is shown in all works on Annuities Certain. But when we come to the next order, $\overset{2}{\mathrm{V}}_n$, which is equal to $\frac{\overset{1}{\mathrm{V}}_n - n(1+i)^{-n}}{i}$, where the second term consists of two factors one of which *increases*, while the other *decreases*, and both without limit, the ultimate result is by no means so apparent. A simple application of the Differential Calculus, however, shows that the term in question vanishes, like the corresponding term in the expression for $\overset{1}{\mathrm{V}}_n$. When the numerator and denominator of a fraction both increase without limit, to determine the limit of the value of the fraction, we must substitute for the numerator and denominator their respective differential coefficients, repeating the process, if necessary, until a fraction is obtained of a form exhibiting its ultimate or limiting value. In the case of $n(1+i)^{-n}$, or $\frac{n}{(1+i)^n}$, a single substitution gives $\frac{1}{\log(1+i)\times(1+i)^n}$, which evidently vanishes when n is increased without limit; and hence we conclude that the primitive fraction, $\frac{n}{(1+i)^n}$, does the same. We have then $\overset{2}{\mathrm{V}}_\infty = \frac{\overset{1}{\mathrm{V}}_\infty}{i} = \frac{1}{i^2}$.

In the case of Annuities of the third order we have $\overset{3}{V}_n = \dfrac{\overset{2}{V}_n - n\dfrac{n-1}{2}(1+i)^{-n}}{i}$. Making the requisite substitution in the fraction $\dfrac{\frac{1}{2}n(n-1)}{(1+i)^n}$, we obtain $\dfrac{n-\frac{1}{2}}{\log(1+i).(1+i)^n}$, which however does not answer our purpose, as the numerator and denominator both increase without limit with n. But repeating the process upon the expression last obtained we get $\dfrac{1}{\{\log(1+i)\}^2(1+i)^n}$, which enables us to see by inspection that the limit we are in search of is 0, as in the two preceding cases. We have therefore $\overset{3}{V}_\infty = \dfrac{\overset{2}{V}_\infty}{i} = \dfrac{1}{i^3}$.

Proceeding in the same way with the higher orders of Annuities, we shall find that to determine the limit of the term $\dfrac{n(n-1)..(n-t+2)}{1.2\ldots.t-1}(1+i)^{-n}$,—which is the $(n+1)$th payment of the tth order,—we should have to perform $t-1$ substitutions, obtaining for the final result $\dfrac{1}{\{\log(1+i)\}^{t-1}(1+i)^n}$. Hence it appears that the law holds good for any finite number of orders, as the expression last given vanishes when n *does*, and t *does not*, increase without limit; and we have therefore for the value of the perpetuity:

$$\frac{u_1}{i} + \frac{\Delta u_1}{i^2} + \frac{\Delta^2 u_1}{i^3} + \ldots$$

which is identical with Mr. Gray's formula for the same problem, as given in a foot-note, page 93.

When the differences of the series of payments do not vanish after a given finite number of orders, the formulæ have to be used with caution; as if the number of payments be great, the error caused by the neglected differences may be considerable. Let us take the following case:

First payment	1·000000				
		+·010000			
Second ,,	1·010000		+·000100		
		+·010100		+·000001	
Third ,,	1·020100		+·000101		+·000000
		+·010201		+ . .	
Fourth ,,	1·030301		+ . .		
		+ . .			
. . .	. . .				

Now these four terms are in exact geometrical progression, the ratio being 1·01, and we see that the differences diminish with

considerable rapidity,—the fourth having no significant figure in the sixth decimal place. Let it be required to find the value of the perpetuity, at 5 per Cent., by means of the formula $\frac{u_1}{i}+\frac{\Delta u_1}{i^2}+\frac{\Delta^2 u_1}{i^3}+\frac{\Delta^3 u_1}{i^4}$.

$$\begin{array}{r|l}
\Delta^3 u_1 = \cdot 000001 & \times 20 \left(=\frac{1}{i}\right) \\
\hline
\cdot 000020 & \\
\Delta^2 u_1 = \cdot 000100 & \\
\hline
\cdot 000120 & \times 20 \\
\hline
\cdot 002400 & \\
\Delta u_1 = \cdot 010000 & \\
\hline
\cdot 012400 & \times 20 \\
\hline
\cdot 248000 & \\
u_1 = 1\cdot 000000 & \\
\hline
1\cdot 248000 & \times 20 \\
\hline
24\cdot 960000 &
\end{array}$$

The true value of this perpetuity is easily found, thus: nth payment $=(1\cdot 01)^{n-1}$; present value of same $=(1\cdot 01)^{n-1}.(1\cdot 05)^{-n}$ $=\frac{1}{1\cdot 01}\cdot\left(\frac{1\cdot 05}{1\cdot 01}\right)^{-n}$;—the sum of which expression from 1 to infinity is $\frac{1}{1\cdot 01}\cdot\frac{1}{\frac{1\cdot 05}{1\cdot 01}-1}=\frac{1}{\cdot 04}=25$.* So that although we have used six decimal places in determining the differences, we do not get more than one correct decimal place in the result. This will not be wondered at when it is considered that the coefficient of the fourth difference $\left(\frac{1}{i^5}\right)$—which is not taken into account—amounts to no less than 3,200,000. Of course, however, any required degree of accuracy may be attained by computing a sufficient number of terms to a sufficient number of decimal places;—provided only that the differences of the series of payments diminishes faster than the increase in the coefficients.

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* From this it is seen that the value of the increasing perpetuity is equal to the value of a fixed perpetuity of £1. calculated upon the assumption that the rate of interest is diminished by the rate of increase in the payments. Since the text was written I have received a communication from Mr. H. Mountcastle, of the Northern Assurance Office, pointing out the same result. The proposition is by no means *self-evident*, and it is not *true* for an annuity for a limited term.

I have stated that Baily's treatment of the subject of Increasing Annuities is not in my opinion satisfactory. Indeed when it is considered that his investigation is confined to Annuities whose successive payments are the several powers of the natural numbers, and that he makes no attempt to show that such Annuities are available for the solution of cases where the law of progression is different, it may naturally be asked why Baily should have devoted two chapters of his work to a subject invested with so limited a share of interest. The answer is found in an observation which he makes at the commencement of his investigation, from which it appears that the object was more especially to aid in the computation of life annuities on the principle of De Moivre's hypothesis; an object which the progress of the science of life contingencies has deprived of the importance which it possessed in Baily's eyes.

In Baily's arrangement the nth payment of an annuity of the tth order is n^{t-1},—while in that which I propose to adopt it is $\frac{(n-1)(n-2)\ldots(n-t+1)}{1.2\ \ldots\ .(t-1)}$. The advantage attending this substitution, consists in the fact that we are thereby put in possession of a key to the solution of all cases in which the differences of the payments vanish or become insignificant after a definite number of orders.

I think I have seen it pointed out, but cannot at present say where, that a life annuity may be considered as an annuity certain the successive payments of which are $p_{x|1}$, $p_{x|2}$, $p_{x|3}$, &c. Indeed the remark applies to any series the successive terms of which are discounted. It has therefore occurred to me—although I have made no attempt to verify the suggestion—that the preceding method might possibly be of use occasionally in the solution of certain problems in Life Contingencies.

The curious result pointed out by Mr. Makeham in the footnote on p. 198 may be thus illustrated. Let the successive annual payments of any benefit be $1+j$, $(1+j)^2$, $(1+j)^3$, ...; then following the course of reasoning used on p. 146, we see that the value of the benefit, estimated at a rate of interest i, is the same as that of a uniform benefit of £1, calculated at a rate of interest, I, equal to $\frac{i-j}{1+j}$. In the case of a perpetual annuity certain, the value will be $\frac{1}{\mathrm{I}}$, or $\frac{1+j}{i-j}$; and is therefore equal to that of a perpetual annuity certain of $1+j$, calculated at a rate of interest $i-j$. Ed. *J. I. A.*

On the arrangement of Commutation, or D *and* N, *Tables. By* MR. JAMES CHISHOLM, *of the North British and Mercantile Insurance Company, Edinburgh.*

[Read before the Institute, 27th January, 1868.]

THERE has been no more valuable improvement in the science of Assurance calculations than that effected by the invention of the Commutation or D and N System of calculation;—a method which, as is now universally known through the writings of Baily, Professor De Morgan and Mr. Gray, we owe, in this country, to the genius of Mr. George Barrett, of Petworth, Sussex. It appeared at first in an incomplete and rather inconvenient form, but various important modifications and extensions of the principle of the method were some time afterwards made; and it was practically applied to the calculation of Life Contingencies by the late eminent actuary Mr. Griffith Davies, with whose name it is now generally associated.

Several works containing voluminous Tables on the Commutation principle have since been published, and actuarial calculations are now greatly facilitated by their use. A certain degree of confusion, however, has nearly from the first surrounded the system, which is especially perplexing to the student, and has, no doubt, deterred many from using the formulæ to that extent which their importance and convenience would warrant.

This confusion has arisen, partly, but to a very slight extent, from a difference between Mr. Davies and Professor De Morgan in tabulating the Assurance values in Column C, and partly from the change introduced by Dr. Farr in 1844 in the system of summation of the columns, or from both causes combined.

The change referred to was from the Terminal to the Initial System of summation; using the latter phrase, as proposed by Professor De Morgan in a paper in Volume X. of the *Journal* "On the Forms under which Barrett's method is presented, &c.," to denote the plan introduced by Dr. Farr; and the former for that followed by Davies in summing the fundamental columns.

On a comparison of the formulæ for the values of an Annuity and an Assurance on these two systems, a want of symmetry is at once apparent. On the Initial System they are $\frac{N_{x+1}}{\mathrm{D}_x}$* and $\frac{\mathbb{M}_x}{\mathrm{D}_x}$, the former signifying the value of an Annuity of which the first pay-

* An explanation of the styles of capital employed in this paper will be found in the postscript.

ment is due at the *end* of a year, the latter the value of an Assurance payable at the *end* of the year of death; while Davies's formulæ for the same values are $\frac{N_x}{D_x}$ and $\frac{M_x}{D_x}$ respectively.

If Davies's formulæ are right, we would naturally expect them, when transferred to the Initial System, to appear as $\frac{N_{x+1}}{D_x}$ and $\frac{M_{x+1}}{D_x}$; and this consideration points us to the M column, or to the C column from which it proceeded, as the places where we are likely to find that the want of symmetry has arisen.

We propose, in this paper, to inquire generally into the causes of the confusion already alluded to, and in doing this it is hoped that some part of it will be cleared away.

The most satisfactory mode of proceeding will be to construct a Commutation Table, on proper principles, as a basis of comparison, to note the differences between it and the Tables of Davies and Dr. Farr, and to examine whether they are essential, and whether they relate to the principle of construction, or not.

At the outset, however, we are led to ask if our Tables of Mortality, on which of course the D and N Tables are founded, are properly constructed; and to this point, therefore, we shall first turn our attention.

In tabulating the numbers living and dying in a Table of Mortality, the practice is to put the number dying in any year of life in a line with the number out of whom they die,—that is, opposite the last age which they have attained. For instance, opposite or on a line with x, l_x signifies the number alive at age x, and $d_x(=l_x-l_{x+1})$ the number dying out of the l_x in the year of life after age x, or in the $(x+1)$th year of age.

Now, on consideration, it will be acknowledged, we think, that it would be more correct to put d_x, the number who die in the $(x+1)$th year of age, opposite age $x+1$. For the d_x and l_{x+1} may be conceived to have sprung from l_x as their origin; both having attained age x, the former *die in* their $(x+1)$th year, the latter *live over* it. The Table, in fact, is merely a list of the numbers of those who, out of a certain initial number, *go forward* to *die in* and *live over* each successive year of life. In this view of the case, therefore, the numbers that have passed a particular age naturally go together, and are accounted for or registered at the next higher age as having *died in* or *lived over* the preceding year.

The case is still stronger when it is considered that for all purposes of calculation, unless otherwise expressed, the (l_x-l_{x+1})

deaths are held to have taken place at the *end* of the $(x+1)$th year of life; the Assurance claims arising under them are then payable, and the deaths may thus be held to have occurred at age $x+1$, and, consequently, may be symbolised at that age by δ_{x+1}.

According to this plan of formation the Table would stand thus:—

Carlisle Mortality.

Age.	No. alive.	No. dying in the preceding year.
x.	$\Sigma\delta_{x+1}=l_x$.	δ_x.
0	10000	
1	8461	1539
2	7779	682
3	7274	505
4	6998	276
..	....	...
101	7	2
102	5	2
103	3	2
104	1	2
105		1

where $l_x=\delta_{x+1}+l_{x+1}$ and $\Sigma\delta_{x+1}=\delta_{x+1}+\delta_{x+2}+\delta_{x+3}+\&c.=l_x$.

The Greek letter δ is adopted to represent the number of deaths in this arrangement of the Table to prevent confusion.

The idea of tabulating d_x as now proposed seems to have been in the mind of Griffith Davies, when, in his *Treatise on Annuities* (p. 94) he says:—

"By the Table * * * * we find that at the town of "Northampton, from 1735 to 1780, out of 11650 children all "born alive,

"8650	attained the age of	1 year,	and 3000	died during the	1st year
"7283	. . . . .	2 years, .	1367	. . . .	2nd . .
"6781	. . . . .	3 . . .	502	. . . .	3rd . .
"6446	. . . . .	4 . . .	335	. . . .	4th . .

"and so on, as in the Register known by the name of the North- "ampton Table."—and then follows the Table as ordinarily constructed.

Further on (page 238), the same arrangement as is given above is repeated; and, at the same place, in explaining the construction of the C column, or rather the P column as he called it, he says:—

"Again, referring to the column expressing the number of "claims to be made on the supposed fund at the end of the 1st, "2nd, 3rd, &c. year, it is manifest that if only £1 were paid to

"each claimant, the present value of the claims becoming due at
"the end of the

"1st year would be	3000 × ·970874 =	2912·622
"2nd	1367 × ·942596 =	1288·529
"3rd	502 × ·915142 =	459·401
"4th	335 × ·888487 =	297·643

"and so on as in column P of the subjoined Table." In this column P (see p. 239) the value 2912·622 is put opposite age 1, 1288·529 opposite age 2, and so with the other values.

As shown above, then, we see that in *some* instances in his work, Davies arranged the numbers of living and dying as now suggested. But whether this was his uniform practice, is not of any consequence. It is enough that in constructing column P, he acted on the footing of this arrangement of the Mortality Table. It will be of importance to bear this in mind when his form of Commutation Table comes to be considered.

The same idea of tabulating d_x is expressed in Jones (*Treatise on Annuities*, Arts. 189 and 192) when he calls the number opposite age x in the column of decrements in a Mortality Table d_{x+1}, indicating that for the purpose at least for which he then uses the number, it would have been more suitably placed at age $x+1$.

We are now in a position to go on with the construction of a Commutation Table; and in doing so we shall adopt the Initial System of summation, it being, as appears to us, the most natural one, and consequently the best for the purpose for which the Table is to be used.

There is nothing unusual in the formation of the Annuity columns. In the Mortality Table we have at age 0 the number l_0, and this multiplied by v^0 gives the first value in the D column, which we place at age 0 and call D_0. Thus $D_0=l_0v^0$, $D_1=l_1v^1$, $D_2=l_2v^2$, &c., as appears in all Commutation Tables.

The N column is thus composed:—

$$N_0=D_0+D_1+D_2+\&c. \text{ to the end of life.}$$
$$N_1=D_1+D_2+D_3+\&c. \qquad \text{do.}$$

and similarly with the S column,

$$S_0=N_0+N_1+N_2+\&c. \text{ to the end of life.}$$
$$S_1=N_1+N_2+N_3+\&c. \qquad \text{do.}$$

The fundamental Assurance column C is not the same as in most of the other Tables, and special attention must therefore be given to its formation. It is formed in the same way from the

Mortality Table. Thus the first number in the column of decrements occurs at age 1. We accordingly multiply δ_1 by v^1 and get the first value in the C column, which we naturally place at age 1. We have then $C_0=0$, $C_1=\delta_1 v^1$, $C_2=\delta_2 v^2$, and so on.

The M and R columns are constructed from C and M respectively, just as the corresponding Annuity columns N and S were formed from D and N:

$$M_1=C_1+C_2+C_3+\&c. \text{ to the end of life.}$$
$$M_2=C_2+C_3+C_4+\&c. \quad \text{do.}$$

and

$$R_1=M_1+M_2+M_3+\&c. \quad \text{do.}$$
$$R_2=M_2+M_3+M_4+\&c. \quad \text{do.}$$

For the sake of clearness, the following Specimen Table is appended:—

Commutation Table for Single Lives. Carlisle 3 per cent.

TABLE I.—*Initial arrangement.*

Age.	$l_x \times v^x$	$\Sigma D_x = D_x + D_{x+1} + \&c.$	$\Sigma N_x = N_x + N_{x+1} + \&c.$	$\delta_x \times v^x$	$\Sigma C_x = C_x + C_{x+1} + \&c.$	$\Sigma M_x = M_x + M_{x+1} + \&c.$
x	D_x	N_x	S_x	C_x	M_x	R_x
0	10000·0000	183198·2348	3885247·9466			
1	8214·5631	173198·2348	3702049·7118	1494·1750	4664·1296	70035·6729
2	7332·4536	164983·6717	3528851·4770	642·8506	3169·9546	65371·5433
3	6656·7404	157651·2181	3363867·8053	462·1465	2527·1040	62201·5887
4	6217·6324	150994·4777	3206216·5872	245·2225	2064·9575	59674·4847
⋮	⋮	⋮	⋮	⋮	⋮	⋮
101	·3536	·7879	1·4576	·1010	·4317	1·1772
102	·2452	·4343	·6696	·0981	·3307	·7455
103	·1429	·1891	·2353	·0952	·2326	·4148
104	·0462	·0462	·0462	·0925	·1373	·1822
105				·0449	·0449	·0449

The Expression here for the value of an Assurance payable at the *end of the year of death* will be $\frac{M_{x+1}}{D_x}$, just as $\frac{N_{x+1}}{D_x}$ expresses the value of an Annuity payable at the *end of each year of life.*

These expressions are symmetrical, and, if it be admitted that the Table from which they are taken is properly constructed, the analogy between the Annuity and Assurance formulæ may be held to be established.

It may be remarked that the analogy does not hold throughout. There is no corresponding Assurance value to that of an Annuity-due; or, if it were possible to deal practically with such values,

the formula deduced for them would not be symmetrical with the Annuity formula.

But this is an erroneous argument, inasmuch as the use of the word Annuity in this sense is improper. Interpreted, an Annuity-due of £1 means a *present payment* of £1, and an Annuity—(the only legitimate sense in which the word can be used)—of £1 thereafter, and the phrase has only been adopted for the sake of convenience, as such values as it indicates are often required. (*On this point see Prof. De Morgan's paper in Vol. X., pp.* 301 *and* 302.)

In comparing the Table we have now constructed with the others in existence, the first that presents itself for consideration is Davies's.

His Table is as below :—

Commutation Table for Single Lives. Carlisle 3 *per cent.*

TABLE II.—*Davies's or Terminal Arrangement.*

Age.	$l_x \times v^x$	ΣD_{x+1} $= D_{x+1} + D_{x+2} +$ &c.	ΣN_x $= N_x + N_{x+1} +$ &c.	$\delta_x \times v^x$	ΣC_{x+1} $= C_{x+1} + C_{x+2} +$ &c.	ΣM_x $= M_x + M_{x+1}$ + &c.
x	D_x	N_x	S_x	C_x	M_x	R_x
		[183198·2348]	[3885247·9466]			
0	10000·0000	173198·2348	3702049·7118		4664·1296	70035·6729
1	8214·5631	164983·6717	3528851·4770	1494·1750	3169·9546	65371·5433
2	7332·4536	157651·2181	3363867·8053	642·8506	2527·1040	62201·5887
3	6656·7404	150994·4777	3206216·5872	462·1465	2064·9575	59674·4847
4	6217·6324	144776·8453	3055222·1095	245·2225	1819·7350	57609·5272
⋮	⋮	⋮	⋮	⋮	⋮	⋮
101	·3536	·4343	·6696	·1010	·3307	·7455
102	·2452	·1891	·2353	·0981	·2326	·4148
103	·1429	·0462	·0462	·0952	·1373	·1822
104	·0462			·0925	·0449	·0449
105				·0449		

We here observe that the fundamental columns D and C occupy the same positions as in Table I., and, bearing in mind how, in that Table, these were formed from the numbers in the Mortality Table, and remembering also what was before stated, that Davies acted on the understanding that the Mortality Table was in the same form as the one on which we founded in constructing Table I., this is just what we should have expected.

The only difference, then, between Tables I. and II. is this, that in the latter, the N, S and M, R columns are raised one step, whereas in the former, the values at the youngest age in the derived columns are put in a line with the values at the youngest age in

the fundamental columns; or, in other words, Davies has adopted the Terminal System of summation, while the Initial is used in Table I. This does not affect the principle of construction, and we consequently find that the expressions for the value of an Annuity and of an Assurance are symmetrical, as they were before, the one being $\frac{N_x}{D_x}$, and the other $\frac{M_x}{D_x}$.

Jones in his *Treatise on Annuities* followed Davies exactly in his arrangement of the columns, but it is easy to miss noticing this owing to his not having stated specifically how he derived column M from column C, and to his not having printed the latter along with the other columns.

In Art. 116, however, the following explanation is given:—

> "Column M is formed by multiplying the decrements opposite each age "in Table I. [Table of Mortality] by the present value of £1 due the same "number of years as the age increased by unity, and taking the successive "sums from the extremity of life, as in the formation of column N from "the numbers in column D."

Paying attention to the statement here that column M is formed "*as in the formation of column* N from the numbers in column D," we must agree that column M would have appeared *one step above* column C if the latter had been printed; and, noting the position M occupies in Jones's Commutation Table, we find that the first value in C must have been at age 1.

Again, in deducing the value of an Assurance by Davies's method (Art. 189) he brings out the formula

$$A_x = \frac{vd_{x+1} + v^2 d_{x+2} + v^3 d_{x+3} + \&c.}{l_x}$$

or

$$= \frac{v^{x+1}d_{x+1} + v^{x+2}d_{x+2} + v^{x+3}d_{x+3} + \&c.}{v^x l_x}$$

calling it equal to $\frac{M_x}{D_x}$, and by so doing, he shows that the M column has been raised above the C column one step. For, independently of what has been shown above, it is *most natural* to suppose that he would place $v^{x+1}d_{x+1}$ opposite age $x+1$ in column C, in which case the expression for the value of an Assurance would have been $\frac{M_{x+1}}{D_x}$ had the M column not been raised.

Intimately connected with the labours of Griffith Davies are those of Professor De Morgan, who gave to the profession two valuable papers which first appeared in the *Companion to the Almanac* for 1840 and 1842.

With the exception of the C column, the Commutation Table there given is the same as Davies's, and it is not necessary to reproduce it here. The exception, however, is one which we must notice. Professor De Morgan makes $(l_0-l_1)v=\mathrm{C}_0$, $(l_1-l_2)v^2=\mathrm{C}_1$, &c., that is to say, he raises the C column one step higher than it is in Davies's arrangement or in Table I. But he did not alter the position of the M and R columns; and these, as well as the N and S columns are in the same position relatively to each other and to D as with Davies.

The only effects of the alteration are that the C column occupies what we venture to think is an anomalous position, and that the Table presents the rather curious condition of having the Annuity division summed on the Terminal and the Assurance division on the Initial System. The formulæ themselves are affected only to a very trifling extent. The difference between them and Davies's formulæ appears only in Benefits on the C column, and this is so very little required that no confusion in using Professor De Morgan's papers is practically experienced.

It may be interesting, however, as well as useful, to append some of the "formulæ and additional notation" given by him in these papers, making the necessary corrections on the supposition that column C is placed as Davies had it.

The following may be compared with those given on page 333 of Vol. XII of this *Journal*:—*

I. $\mathrm{N}_x=\mathrm{D}_{x+1}+\mathrm{D}_{x+2}+\mathrm{D}_{x+3}+\ldots$ to the end of life.

$\mathrm{M}_x=\mathrm{C}_{x+1}+\mathrm{C}_{x+2}+\mathrm{C}_{x+3}+\ldots$ do.

$\mathrm{S}_x=\mathrm{N}_x+\mathrm{N}_{x+1}+\mathrm{N}_{x+2}+\ldots=\mathrm{D}_{x+1}+2\mathrm{D}_{x+2}+3\mathrm{D}_{x+3}+\ldots$

$\mathrm{R}_x=\mathrm{M}_x+\mathrm{M}_{x+1}+\mathrm{M}_{x+2}+\ldots=\mathrm{C}_{x+1}+2\mathrm{C}_{x+2}+3\mathrm{C}_{x+3}+\ldots$

II. $\mathrm{N}_x-\mathrm{N}_{x+1}=\mathrm{D}_{x+1}$ $\qquad$ $\mathrm{S}_x-\mathrm{S}_{x+1}=\mathrm{N}_x$

$\mathrm{M}_x-\mathrm{M}_{x+1}=\mathrm{C}_{x+1}$ $\qquad$ $\mathrm{R}_x-\mathrm{R}_{x+1}=\mathrm{M}_x$

* * * * * *

V. $\mathrm{D}_x+\ldots+\mathrm{D}_y=\mathrm{N}_{x-1,\,y-x+1}$

$\mathrm{C}_x+\ldots+\mathrm{C}_y=\mathrm{M}_{x-1,\,y-x+1}$

$\mathrm{D}_x+\ldots+(y-x+1)\mathrm{D}_y=\mathrm{S}_{x-1,\,y-x+2}$

$\mathrm{C}_x+\ldots+(y-x+1)\mathrm{C}_y=\mathrm{R}_{x-1,\,y-x+2}$

* * * * * *

VIII. $\mathrm{C}_x=v\mathrm{D}_{x-1}-\mathrm{D}_x$

$\mathrm{M}_x=v\mathrm{N}_{x-1}-\mathrm{N}_x$

$\mathrm{R}_x=v\mathrm{S}_{x-1}-\mathrm{S}_x$

* Professor De Morgan's papers have been reprinted in this *Journal* in Volume XII., p. 328, and Vol. XIII., p. 129. Mr. Gray's papers, afterwards referred to, are reprinted in Volume X.

It should be observed that the preceding remarks apply also to the papers by Mr. Gray which originally appeared in the Mechanics' Magazine for 1842, and which are intended as an Introduction to those of Professor De Morgan.

There only remains now to be considered the Table of Dr. Farr. For the purpose of comparison the following Specimen of his arrangement is annexed.

Commutation Table for Single Lives. Carlisle 3 *per cent.*

TABLE III.—*Dr. Farr's arrangement.*

Age.	$l_x \times v^x$	$\Sigma \mathrm{D}_x = \mathrm{D}_x + \mathrm{D}_{x+1} + \&c.$	$\Sigma N_x = N_x + N_{x+1} + \&c.$	$d_x \times v^{x+1}$	$\Sigma \mathrm{C}_x = \mathrm{C}_x + \mathrm{C}_{x+1} + \&c.$	$\Sigma \mathrm{M}_x = \mathrm{M}_x + \mathrm{M}_{x+1} + \&c.$
	D_x	N_x	S_x	C_x	M_x	R_x
0	10000·0000	183198·2348	3885247·9466	1494·1750	4664·1296	70035·6729
1	8214·5631	173198·2348	3702049·7118	642·8506	3169·9546	65371·5433
2	7332·4536	164983·6717	3528851·4770	462·1465	2527·1040	62201·5887
3	6656·7404	157651·2181	3363867·8053	24[illegible]·2225	2064·9575	59674·4847
4	6217·6324	150994·4777	3206216·5872	173·3844	1819·7350	57609·5272
⋮	⋮	⋮	⋮	⋮	⋮	⋮
101	·3536	·7879	1·4576	·0981	·3307	·7455
102	·2452	·4343	·6696	·0952	·2326	·4148
103	·1429	·1891	·2353	·0925	·1373	·1822
104	·0462	·0462	·0462	·0449	·0449	·0449
105					·	

By this Table the expression for the value of an Assurance is $\frac{\mathrm{M}_x}{\mathrm{D}_x}$, and of an Annuity $\frac{N_{x+1}}{\mathrm{D}_x}$, as pointed out in the beginning of this paper. These, unlike the same formulæ under Tables I. and II., are not symmetrical, and from what has been already said the reason of this will be evident.

When Dr. Farr introduced the change from the Terminal to the Initial System of summation,* he adopted Professor De Morgan's $\mathbb{C}$ column, and with it the whole of his Assurance division, which, as was before stated, was already summed on the Initial principle, and did not therefore require any change. The only alteration on Professor De Morgan's Table was that columns N and S were brought down to a level with D. The change thus affected only the Annuity half of the Table, and this accounts for the want of symmetry in Dr. Farr's formulæ.

* See Part II. of the Appendix to the Registrar-General's Report for 1844, pages 526 and 527 and 592 and 593. Compare also Introduction to the English Life Table No. 3, page 119 *et seq.*

Besides the confusion, then, of having two different systems in existence as the basis of our formulæ, we have the additional confusion that in the one system (Davies's) the Assurance and Annuity Formulæ are counterparts of each other, while in the other system (Dr. Farr's) they are not; and this disappearance of the analogy between the two divisions of the Table on Dr. Farr's system certainly renders the formulæ more difficult of remembrance.

It should not be forgotten, however, that when Professor De Morgan contributed his papers before referred to, and when Dr. Farr first introduced his change, the posthumous work of Mr. Davies from which we have quoted was not published; and that the Tract by him on Life Contingencies, which was at the time, the principal work on the subject, did not contain so full an exposition of his views as his later Treatise, and was besides very scarce.

On referring to this "Tract" we find that the column P is not tabulated along with the other columns; and that all the explanation that is given of the formation of M is contained in a short paragraph expressed in almost the very words employed by Jones for the same purpose, as quoted at page 206.

It will be seen to be important to quote Davies's own words. He says (p. 36):—

> "The numbers in column M have been deduced by multiplying the "decrements given opposite each age in Table X. [*Table of Mortality*] "by the present value of £1 due as many years hence as are equal to that "age increased by unity; and then, beginning at the extremity of life, and "taking the successive sums of the results, the same as in the formation of "column N from D, or S from N."

It will be observed that Davies directs the successive sums to be taken, "the same as in the formation of column N from D, *or* S *from* N." Now, in forming N from D, that column was raised one step, while S was kept on the same level as N, so that from this passage we could not tell whether Davies meant the column from which M was formed to be one step below, or on a level with M.

In the absence of more precise information, therefore, one would be quite at liberty, while professing to follow Davies, to place the C column in either position. Thus, Professor De Morgan fixed C with the first value at age 0, and this position of the column was confirmed by Dr. Farr; but, as it turned out afterwards, (on the publication of Davies's Treatise by his Executors in 1856) this was not the place Davies meant C to occupy. We have here,

then, the explanation of the manner in which C ever came to hold two positions as it does now. The small extent to which Davies's work is used in practice, and the unimportance of the C column in itself, have no doubt contributed to this fact being overlooked, apparently, until now.

To summarise, Davies placed the first value in D at age 0, and the first value in P or C at age 1, and summed these columns on the Terminal System. Professor De Morgan placed the first values in both columns D and C at age 0, summing the former on the Terminal, and the latter on the Initial System. Dr. Farr placed the first values in D and C at age 0 following Professor De Morgan, and summed both columns on the Initial System. And lastly, in the specimen Table constructed for this paper N and M are formed by summing D and C, respectively, on the Initial System; these last-named columns occupying the position assigned to them by Davies.

If it be agreed that this is their proper position, Table I. will present the correct form of a Commutation Table on the Initial, and Table II. the correct form on the Terminal system of summation.

The principal writers who have followed Davies are Jones in his *Treatise on Annuities* (1844), Gray, Smith and Orchard in their *Three per cent Assurance and Annuity Tables* (1851), and Brown in his *Report* on the *Madras Military Fund* (1863). In the two last-named works no reference at all appears to be made to C, so that it is impossible to say whether the authors have followed Davies in the position he gave to the column, or not. Professor De Morgan in his two papers in the *Companion to the Almanac* and Mr. Gray in his Introduction to them may also be said to have followed Davies, for the displacement there of the C column has little practical effect.

Dr. Farr's method has been followed in those of the Registrar-General's Reports since 1844 where the subject has been touched upon, and by Chisholm in his *Commutation Tables* (1858), while other writers in the *Journal of the Institute* and elsewhere have adopted that method which seemed most convenient to them. Such appears to be the position in which we are now placed with reference to the Commutation System. The settlement of the question which form of it is to be adopted as the standard will be a great advantage. Until then, we must make the best of the forms we have got, and endeavour to understand clearly the symbols we have to work with. If the present paper prove at

all beneficial in this last respect the object of the writer will have been fully served; and, in conclusion, it may be remarked that it will be found convenient, in using the Tables, to adopt some distinctive lettering for the different forms, such as that shown in the Postscript.

NOTE.—Since the foregoing was written it has been observed that the registration of the numbers living and dying, which occurs at the margin of the "Student's Table," compiled in 1849 by Mr. W. T. Thomson, Manager of the Standard Assurance Company, is made in the same way as suggested in this paper. In the arrangement, also, of *all* the columns Mr. Thomson has exactly followed Davies, and in his work entitled "Actuarial Tables," where the Student's Table is referred to and explained, he has consistently carried out his plans by giving the values of an Assurance at all ages from 1 to 105 in place of from 0 to 104.

POSTSCRIPT.

The following statement is appended in explanation of the different styles of lettering employed in the preceding paper. The columns in Davies's arrangement, as the correct form of Table on the Terminal System, were first lettered; next the columns as in Table I. in the paper, as the correct Initial arrangement, and lastly the columns as in Dr. Farr's arrangement.

Terminal arrangement.

Elements of Column.	Name of Column.
$v^x l_x$	D_x
$D_{x+1}+D_{x+2}+D_{x+3}+$&c.	N_x
$N_x+N_{x+1}+N_{x+2}+$&c.	S_x
$v_x(l_{x-1}-l_x)=v^x\delta_x=v^x d_{x-1}$ . . .	C_x
$C_{x+1}+C_{x+2}+C_{x+3}+$&c.	M_x
$M_x+M_{x+1}+M_{x+2}+$&c.	R_x

This is the original tabulation of the columns; and Roman Capitals are used for the headings in every case. In Professor De Morgan's modification of this arrangement the heading of the C column will be printed in open capitals; all the other characters as they are above.

Initial arrangement.

Elements of Column.	Name of Column.
$v^x l_x$	D_x
$\mathrm{D}_x+\mathrm{D}_{x+1}+\mathrm{D}_{x+2}+$&c.	N_x
$N_x+N_{x+1}+N_{x+2}+$&c.	S_x
$v^x(l_{x-1}-l_x)=v^x\delta_x=v^x d_{x-1}$ . . .	C_x
$\mathrm{C}_x+\mathrm{C}_{x+1}+\mathrm{C}_{x+2}+$&c.	M_x
$M_x+M_{x+1}+M_{x+2}+$&c.	R_x

In this arrangement, Italic Capitals have been used as the distinctive headings of the columns; but as the D and C columns are the same as Davies's the Roman lettering in these two cases is retained.

Dr. Farr's arrangement.

Elements of Column.	Name of Column.
$v^x l_x$	D_x
$\mathrm{D}_x+\mathrm{D}_{x+1}+\mathrm{D}_{x+2}+$&c.	N_x
$N_x+N_{x+1}+N_{x+2}+$&c.	S_x
$v^{x+1}(l_x-l_{x+1})=v^{x+1}\delta_{x+1}=v^{x+1}d_x$.	$\mathbb{C}_x$
$\mathbb{C}_x+\mathbb{C}_{x+1}+\mathbb{C}_{x+2}+$&c.	$\mathbb{M}_x$
$\mathbb{M}_x+\mathbb{M}_{x+1}+\mathbb{M}_{x+2}+$&c. . . .	$\mathbb{R}_x$

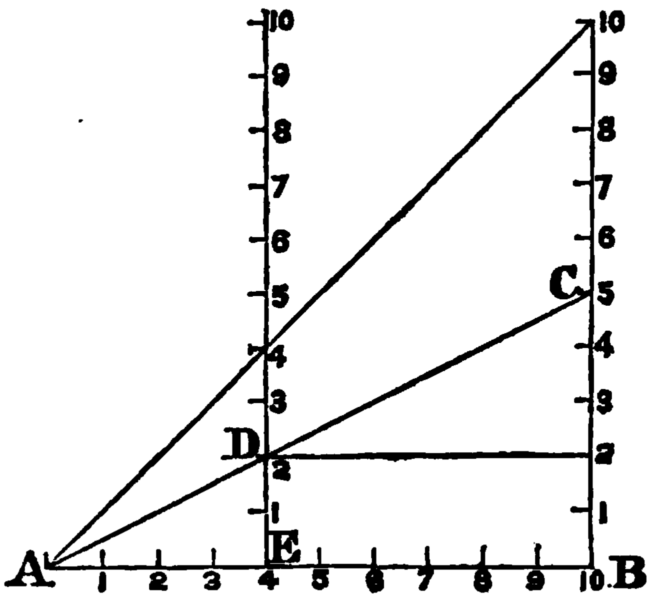
As suggested by Mr. Sprague in last number of the *Journal,* Open Capitals have been used here as the distinctive headings of the columns; but, as the D column is the same as Davies's the Roman lettering for it is still retained. The N and S columns being the same as in the previous arrangement, retain the Italic Capitals as their headings.

Memoir on Instrument for furnishing the D *numbers, to four figures each, in Two Joint Life Annuity Tables, on any basis. By* JARDINE HENRY, *Author of "The Government Annuity Tables," "The Government Life Annuity Commutation Tables," "The Hand Book for Life Assurers," &c. &c.*

[Read before the Institute of Actuaries on 25th November, 1867, and printed in abstract by order of the Council.]

IN this paper, Mr. Henry describes an instrument by the aid of which he has actually obtained the values (to four figures) of many thousand products, being the D values in the various tables he has published. The principle of the instrument, which he claims as

entirely his own invention, may be briefly described as follows:—



Let ABC be a right-angled triangle, of which AB is the base and AC the hypothenuse, and from any point D in the hypothenuse let fall DE perpendicular to AB; then from the similar triangles ADE, ABC, we have

$$DE : AE :: BC : AB$$

$$\text{and } DE = \frac{AE \times BC}{AB}$$

Now if AB be taken as equal to 1, 10, 100, 1000, or 10,000, we may say, disregarding the position of the decimal point, that $DE = AE \times BC$. The instrument used by Mr. Henry consists of a brass right-angled triangle with the two legs equal, each 75 inches long, and having each of the three sides divided into 10,000 equal parts. There is also a moveable T similarly graduated which slides along the triangle, and is to be fixed, where required, parallel to the altitude BC. Then if we require to find the product of two numbers, each consisting of four figures, or less, the one of them is set off on the altitude of the instrument BC, and the line AC is drawn. Then the other number is set off along the base of the instrument, being represented by AE; and the upright T being then fixed at E, and cutting the line AC in D, the length DE is read off on the T, and gives the product of the two given numbers represented by AE and BC.

To keep the T in its place on the face of the Triangle, a copper wire may be passed across the Triangle, permitting the T to slide, but not to fall forward.

The interior of the instrument is filled up with a Zinc plate, and on this is pasted a sheet of paper sufficiently large to cover the whole, and left until it is thoroughly dried.

In practice the T is *clamped* with the screw clamp provided for that purpose—a small square of paper in several folds inserted under the screw of the clamp, and also in the opposite side of the instrument, enabling the clamp to hold firmly.

In order to form a table of joint-life annuities, Mr. Henry sets off on the altitude of the instrument the numbers living at all ages from birth to the extremity of life,—say, for 100 ages, and actually draws the various lines from the angle A to these 100 divisions. The T is then fixed at the point E, such that $AE = l_x v^x$, and reading

off the points where it cuts the various lines drawn, we get a series of products of the form

$$l_y l_x v^x,\ l_{y+1} l_x v^x,\ l_{y+2} l_x v^x,\ \&c.$$

In the same way, setting the T for all the possible values of $l_x v^x$, and reading off the intersections with the same lines, we get all the products which constitute the D numbers in a complete table of two joint lives.

The author proceeds:—These products being taken down on paper, horizontally, furnish a set of numbers which, along with other similar horizontal sets of numbers thus obtained, make the D Columns in Commutation Tables of Joint lives—the addition of them, stage by stage, from the oldest age downwards to the youngest, gives the N Columns.

Supposing that an Instrument can be made of sufficient exactness to show the products correct to the fourth figure of each product—it will be found, on trial, that such numbers, added up, are sufficient to furnish the values of Annuities as exactly as could be done by using five figures and Milne's process, and that for all practical purposes, the latter, and consequently the former also, is as good as Milne's process using 7 figures—in fact, with a medium accuracy in the value of an Annuity to within 2 in the fourth figure of decimals of one pound—one-fifth of one farthing, the error being limited to 1*d.* or 1½*d.*, at most, in any individual case of the value of a Life Annuity of £1 per annum.

As showing, by experiment, the comparative accuracy of taking only four figures for products, a trial has been made with two Tables, calculated by Barrett's method using five figures in the products, which must necessarily render the values exact to three places of Decimals, besides the integers.

These Tables were, Male, Single Life, 3 per cent, and Female, Single Life, 5 per cent, both on the Government Life Annuity basis. The average difference of each value calculated with four figures to the product, was, for both Tables, ·0002, viz., 2 in the fourth place of decimals, or one-fifth of one farthing. For Joint Lives where the values are necessarily diminished compared to Single Lives—the average difference must be less than the above.

Any one acquainted with the amount of difference in the best observations of human mortality will be aware that the differences in such Tables extend to one year's purchase of an annuity, and it may thus be inferred that a greater degree of accuracy than that attained by the Instrument is, practically, useless.

But the important point of the application is the facility with which the products can be read from the Instrument, by one person, so as to be taken down by another person to dictation. This facility exceeds the power of the quickest writer of figures to keep up with—400 products being, however, easily tabulated in an hour—the power attainable being only limited by the power of expression, which may be held to extend to 900 products per hour. To accomplish this last, by the ordinary process of multiplication, would take about ten times that duration, and even the assistance of Logarithms would only enable greater accuracy to be attained than can be had in the ordinary process of multiplying, without any saving of time compared to the latter process.

But, in accuracy, it is much superior to what the results of actual multiplication would bring out.

The last 1500 products of Female Life, Mean Duration of Two Joint Lives, being for the ages from 0 to 16, were furnished and tabulated at the rate of about 500 products per hour.

In speaking of the above I have proceeded on the assumption that the D numbers are taken down from the Instrument in horizontal line, and in ink.

If, however, the mode of taking the D numbers down vertically, and in pencil, be adopted—leaving out the figures that repeat themselves, thus—

Government Tables—Female Life—2 joint Lives—
Interest 4 per cent.
Multiplier D at age 48, 8132, for ages 20, 19, 18, &c.

D numbers.
5844
900
50
6001
42
85
130
62
92
&c.

the rate of rapidity is much increased, the Author having taken down the D numbers at the rate of 2310 D numbers per hour—or one complete Table of D numbers for Two Joint Lives, at one rate of interest, in two hours. The addition of the numbers being necessarily horizontal, in this mode, may create a difficulty which, however, practice may overcome.

Using ink, 2000 D numbers were taken per hour. In stating these facts, the rate of taking down from the Instrument after the

T is set, is alone taken into account. The setting of the T does take a little time which must be deducted in making a practical estimate. But the 500 products per hour referred to as tabulated included the time taken in setting the Instrument, which practice makes very short.

By such means the writer of this Memoir has calculated the mean duration of Life for Two Joint Lives of Males, Females, and their combinations of Male (Elder) and Female Life, and Male (Younger) and Female Life, forming, [with Tables of Mean Duration, and at Interest of 1, 2, 2½, 3, 3½, 4, 5, 6, 7, 8, 9, and 10 per cent for Single Lives,] the first Volume published of *The Government Life Annuity Commutation Tables.* A fourth part of the next volume, viz., Two Joint Lives D and N Columns for Male (Elder) and Female Life at 3, 3½, 4, and 4½ per cent, has also been, as regards D columns, furnished by the Instrument. Specimens at 3 per cent interest are appended—showing the D numbers as actually furnished by the Instrument in antique type and the corrected numbers in ordinary type.

Without the aid of such an Instrument he would not have attempted the work, which, however, the Instrument places within practical compass.

Should it be thought desirable to have MS. Tables, founded on any particular data, such Tables could be furnished by the Instrument with the greatest facility.

In calculating Joint Life *Annuities* it may be here shortly explained that when any particular rate of Interest is wanted—say 3½ per cent per annum, for any particular age or table, say Government Females age 10, 7685, the number alive at age 10, is multiplied by the value of £1. payable ten years hence at 3½ per cent—7089, and the product of the two, or 5448, is the number on the horizontal and hypothenuse scales at which the T is fixed for that particular set—and similarly for the other ages.

As regards the different Instruments which the Author has practically used, these are *two.* The first was formed of Boxwood for the divided parts, and the interior of the triangle was filled up with seasoned Baltic timber, and, after paper had been put on, ruled, and this gave an accuracy, on the average of the products, not differing by more than 4 in the last figure—the size of this Instrument was 100 inches to the side, and was constructed by Messrs. A. Adie & Son.

The second Instrument is formed of Brass Sides, and filled up in the interior with Zinc plate and Paper pasted upon the zinc, on

which (the paper) the lines are ruled, and it is 75 inches to the side. It gives a nearer approach to accuracy, and was also constructed by Messrs. A. Adie & Son.

The error of the products, with reference to the Tables before referred to, has been checked and corrected by using a Table of Multiples to 10 of the original numbers, (*i.e.*, numbers alive) by which means, accuracy to the last figure has been secured.

It is quite possible, the Author believes, to construct a Single Instrument which will accomplish the furnishing of the products correct to the last figure, but as such Instruments have not hitherto been in use to be constructed, there are mechanical difficulties to be overcome.

The Writer may add that, after exhausting the first series of Numbers, running from age 0 to age 76, in the Government Tables of *Females*, which extend from 9754 to 2098, in *Black* lines, he uses *red* lines, with five times the actual numbers, for the Numbers from 77 to 88, which includes the Numbers from 9650 to 2025 (increased five times) *blue* lines for the subsequent series, 89 to 90, taken as they stand, *i.e.* 327·5 and 236·2—then *black* with *intervals*, for a fourth series, 91 to 94, with five times the actual Numbers, *i.e.* 787·5 to 316·2—*red* with *intervals* for a fifth series, 95 to 96, taken as they stand, viz., 35·50 and 23·70—and *blue* with *intervals* for the sixth series, 97 to 100, taken as 5 times the actual numbers, or 98·50 and 39·50, and lastly at age 101, 3·900, taken as it stands, in a blue and black line with intervals, exhausting the Table of Numbers alive, Females.

In reading from the Instrument when the *red* lines are used, as each line represents $\frac{10}{2}$ times its actual value, the numbers on the T scale must be *doubled*, an operation which practice makes as easy as the reading of the original numbers—the necessity of using five times the number arises from ages after 76 going so far down in the scale as to make the products sink to 3 figures, which is remedied by the process referred to.

It is also expedient occasionally to double the multiplier, and read off the *half* of each result, which practice makes equally easy.

In reading the numbers from the Instrument, a careful eye is required, but a young man with a little experience is quite able to take down the numbers in writing to dictation, or, *vice versâ*, to read the numbers from the Instrument to be taken down by another.

In working the Instrument it is desirable to check the particular age taken down, at intervals of 20 or 30 numbers, as the eye may slip over a particular number (or *line*) and cause misplacement,

which, although it must necessarily be discovered at the end of the particular horizontal set of numbers taken down, is more easily corrected at an earlier stage in the work of taking down the products in a horizontal line.

That the invention is a thoroughly practical one, has been proved to the Author by his own experience, which has embraced calculation, by Logarithms, to the extent of a million and a half of written figures, used in the calculation of *The Government Annuity Tables* published by him in 1859, and the 1st vol. of *The Government Life Annuity Commutation Tables* published last year (1866) extending to 300,000 figures, which would, if wrought by Logarithms, have required 300,000 additional written figures.

As regards the time required to rule the Instrument—it will take about two days. It requires to be done by a person accustomed to accurate ruling, such as the assistant of a Civil Engineer. It is necessary to use, as the ruler, a *straightedge*, or steel square, about 7 or 8 feet long. This last can be obtained from a Civil Engineer. A small magnifying glass is also used, to adjust the straightedge to the point indicating the figure to which the line is to be ruled.

As to the time required to compute or tabulate, a complete set of D numbers—

From 1500 to 2000 D numbers can be dictated and written down horizontally upon paper ruled for the purpose, in one day—and I should say that four days should suffice, provided the parties are not delayed by blunders in dictating, or writing down, the numbers.

The great desideratum in working the Instrument is *accuracy*, because the omission to read a single number (or to observe any one line) puts all the remainder of the numbers of the horizontal series, it may be 90, or 100 numbers, wrong in place. When the Reader and Writer are both accurate, there is nothing more beautiful than the working of the Instrument, from its rapidity and the facility it affords in dealing with large sets of figures.

This is, with reference to the particular Table ruled on the Brass Instrument (Female Life, of the Government Annuity Tables (Henry)), more especially applicable to the black lines, seventy-seven in number. The coloured lines, again, are not so easily worked, because, where a particular adaptation of the Multiplier is needed to avoid carrying the T to a number below 5000 on the horizontal scale of the Instrument, and we double or halve the Multiplier for that purpose—there is then some trouble in working

the coloured lines. Supposing we *double* the multiplier, then we must halve the numbers applicable to the black lines, but the red lines having been ruled to numbers 5 times the amount of the proper numbers in the Table (practically *halved*), then the red lines in our supposed case *read natural*—that is, are taken down as they stand. The blue lines again, having been ruled to the true numbers, the numbers on the T scale marked by them are halved, and so on alternately to the end. When the Multiplier is *halved*, then the black line numbers on the T require to be doubled, while the red line numbers would require to be quadrupled, the blue line numbers doubled, and so on alternately. This, however, can generally be avoided, as is desirable, the quadrupling of numbers being a work which obliges recourse to be had to pencil and paper for correct results.

As observed before, the difficulties are confined to the numbers above age 76, that is, a fourth only of the whole, and by judicious management can generally be avoided.

In order to show the working of the Instrument, I shall, after the Tables showing the Numbers alive at every age, give a set of products derived from the Brass Instrument—then the corrected numbers, being those intended for being printed—after disposing thus of the D numbers, the resulting N numbers for the uncorrected and corrected D numbers, will be shown for every tenth division or part of the whole—finally the money values applicable to each—being Male (Elder) and Female Life, 3 per cent.

Male Life.

Number who complete the age opposite to each.							
Age of A.	Number Alive.	Age of A.	Number Alive.	Age of A.	Number Alive.	Age of A.	Number Alive.

Male Life (continued).

Number who complete the age opposite to each.							
Age of A.	Number Alive. *a.*	Age of A.	Number Alive. *a.*	Age of A.	Number Alive. *a.*	Age of A.	Number Alive. *a.*
53	44223	65	30101	76	13707	87	2353
54	43138	66	28829	77	12422	88	1834
55	42115	67	27374	78	11164	89	1452
56	41091	68	25957	79	9973	90	1084
57	40001	69	24331	80	8736	91	752
58	38942	70	22574	81	7489	92	474
59	37870	71	20875	82	6303	93	406
60	36722	72	19380	83	5323	94	338
61	35441	73	17828	84	4597	95	188
62	34108	74	16276	85	3895	96	94
63	32831	75	14947	86	3070		
64	31393						

Note.—The number of Male Children newly born was derived from the probability of a Male Child newly born surviving one year ($_1 0$) as given by the English Life Table, No. 1.

Female Life.

Number who complete the age opposite to each.							
Age of A.	Number Alive. *a.*	Age of A.	Number Alive. *a.*	Age of A.	Number Alive. *a.*	Age of A.	Number Alive. *a.*
0	97544	26	68271	52	50337	78	17454
1	84610	27	67606	53	49555	79	15753
2	83490	28	66965	54	48736	80	14280
3	82020	29	66334	55	47867	81	12438
4	80563	30	65745	56	46922	82	10730
5	79739	31	65083	57	46041	83	9354
6	79013	32	64309	58	45113	84	7965
7	78230	33	63570	59	44151	85	6842
8	77647	34	62911	60	43316	86	5876
9	77204	35	62347	61	42361	87	4900
10	76853	36	61791	62	41233	88	4050
11	76538	37	61174	63	40177	89	3275
12	76216	38	60450	64	39012	90	2362
13	75860	39	59708	65	37759	91	1575
14	75421	40	59036	66	36523	92	1050
15	74907	41	58270	67	35232	93	869
16	74342	42	57584	68	33907	94	632
17	73755	43	56966	69	32432	95	355
18	73157	44	56319	70	30876	96	237
19	72518	45	55698	71	29331	97	197
20	71875	46	54982	72	27740	98	158
21	71271	47	54254	73	26087	99	118
22	70678	48	53437	74	24419	100	79
23	70076	49	52551	75	22699	101	39
24	69500	50	51703	76	20984		
25	68912	51	50976	77	19300		

Note.—The number of Female Children newly born (of age 0), is derived from the English Life Table, No. 1.

Government Life Annuity Commutation Tables. Male (Elder) and Female Life.

Annuity on Two Joint Lives at 3 per Cent.

Difference of Age 12.

Age of Elder.	From Brass Instrument. Uncorrected. D.	From Brass Instrument. Corrected. D.	Uncorrected. N.	Corrected. N.	1+value of Annuity of £1. Brass Instrument. Uncorrected.	1+value of Annuity of £1. Brass Instrument. Corrected.
96	4388	4384				
95	1060	1060				
94	2252	2252				
93	3234	3232				
92	4462	4462				
91	8032	8038				
90	1324	1324	3271	3272	2·471	2·471
89	2019	2019				
88	2854	2854				
87	4076	4081				
86	5885	5900				
85	8240	8236				
84	1066	1066				
83	1342	1342				
82	1728	1724				
81	2215	2215				
80	2784	2784	11769	11767	4·227	4·227
79	3398	3401				
78	4068	4065				
77	4816	4817				
76	5652	5658				
75	6546	6542				
74	7526	7531				
73	8731	8731				
72	9992	9994				
71	1132	1130				
70	1286	1286	8665	8666	6·738	6·739
69	1457	1457				
68	1632	1632				
67	1808	1808				
66	1997	1997				
65	2185	2185				
64	2384	2384				
63	2601	2604				
62	2824	2822				
61	3070	3070				
60	3331	3331	31954	31956	9·593	9·594
59	3592	3592				
58	3853	3855				
57	4133	4133				
56	4420	4420				
55	4723	4723				

Government Life Annuity Commutation Tables.
Male (Elder) and Female Life.

Annuity on Two Joint Lives at 3 per Cent.

Difference of Age 12 (*continued*).

Age	From Brass Instrument.		Uncorrected.	Corrected.	1+value of Annuity of £1. Brass Instrument.

Government Life Annuity Commutation Tables. Male (Elder) and Female Life.

Annuity on Two Joint Lives at 3 per Cent.

Difference of Age 24.

Age of Elder.	From Brass Instrument. Uncorrected. D.	From Brass Instrument. Corrected. D.	Uncorrected. N.	Corrected. N.	1+value of Annuity of £1. Brass Instrument. Uncorrected.	1+value of Annuity of £1. Brass Instrument. Corrected.
96	1532	1527				
95	3330	3326				
94	6478	6485				
93	8430	8426				
92	1060	1060				
91	1798	1798				
90	2767	2767	7602	7601	2·747	2·747
89	3954	3950				
88	5302	5309				
87	7222	7225				
86	9955	9961				
85	1335	1337				
84	1662	1664				
83	2020	2020				
82	2520	2518				
81	3143	3146				
80	3851	3851	17934	17940	4·657	4·659
79	4620	4620				
78	5426	5426				
77	6318	6318				
76	7298	7300				
75	8308	8305				
74	9442	9440				
73	1083	1083				
72	1233	1233				
71	1391	1389				
70	1568	1568	11209	11207	7·149	7·147
69	1763	1763				
68	1958	1958				
67	2154	2152				
66	2359	2359				
65	2570	2570				
64	2797	2797				
63	3045	3045				
62	3301	3301				
61	3570	3573				
60	3852	3852	38578	38577	10·020	10·015
59	4128	4128				
58	4408	4411				
57	4717	4717				
56	5050	5048				
55	5394	5394				

Government Life Annuity Commutation Tables.
Male (Elder) and Female Life.

ANNUITY ON TWO JOINT LIVES AT 3 PER CENT.

Difference of Age 24 (*continued*).

Age of Elder.	From Brass Instrument. Uncorrected. D.	From Brass Instrument. Corrected. D.	Uncorrected. N.	Corrected. N.	1+value of Annuity of £1. Brass Instrument. Uncorrected.	1+value of Annuity of £1. Brass Instrument. Corrected.
54	5750	5750				
53	6124	6124				
52	6522	6521				
51	6914	6917				
50	7326	7332	94911	94919	12·955	12·946
49	7744	7753				
48	8164	8167				
47	8592	8590				
46	9032	9030				
45	9484	9495				
44	1001	1001				
43	1056	1056				
42	1113	1113				
41	1172	1172				
40	1234	1234	19368	19371	15·695	15·685
39	1301	1301				
38	1369	1369				
37	1439	1439				
36	1509	1509				
35	1578	1577				
34	1647	1647				
33	1724	1725				
32	1808	1808				
31	1903	1903				
30	2006	2005	35652	35654	17·782	17·783
29	2111	2111				
28	2224	2224				
27	2366	2368				
26	2517	2520	44870	44877	17·827	17·808
25	2672	2665				
24	3206	3211	50748	50753	15·829	15·806

From the specimens given, which have been taken at random, and may thus be held as a fair representation of the whole, it will be seen that the Brass Instrument's figures, taken without correction are very close to the corrected results, and, as regards the actual values of annuities, derived from either, there is no appreciable difference, excepting at the two last values of annuities, where it rises to 6*d.*

As regards Table of Diff. of Age 12. The greatest difference is at ages 12 M., 0 F., and is ·025 *plus* on Instrument. This arises mainly from the line on the Brass Instrument ruled to the age ·0, representing the number of Females born, being the worst ruled of all.

This appears from the next age, 13 M., 1 F., when the difference is only ·010 plus on Instrument, being the greatest difference except the above, the average of error being ·0015 plus, or, leaving out the last, ·001 plus, equal to $\frac{1}{4}d.$ on each value, or a little less.

On trying the next Table for ages (older) 90, 80, 70, 60, 50, 40, 30, 26, and 24, the greatest difference appears at 24, being ·023 plus on Instrument, and the average is ·007 plus on Instrument, or about $1\frac{1}{2}d.$

From having made the same calculations by the *Wood* Instrument, which had not the advantage of the Brass Instrument in the coloured lines and increased numbers for these, but was ruled uniformly to the bottom of the Triangle, I am enabled to state that the average error in the first Table is ·023, and the greatest error ·088, both plus. In the Second Table, the average error was ·012, and the greatest error ·088, both minus.

This shows, that even with the imperfection of the Wood Instrument, a tolerable approach was made to accuracy. Had the coloured lines been extended to the Wood Instrument, thereby obtaining four figures in every result, (instead of three at the lower lines,) the approach to accuracy would have been much nearer.

As the matter stands, I think it established that an Instrument could be constructed and ruled, so as to give accurately the four figures required, and thus supersede, in the construction of Two Joint Life Tables, the employment of multiplication, or of logarithms.

If it were practicable to add the numbers side by side, the taking down being made so easy, the calculation would be much lighter, it having been stated, as before, that it is perfectly practicable to take down the D numbers at the rate of 2300 of these in an hour.

The time required to shift the T to each Multiplier has not however been taken into account. In taking down 2300 numbers, we would have about 46 T sets of figures, *i.e.* the average number read from each setting of the T being about 50, there would thus be 46 settings of the T—and the time required to set and clamp the T at the bottom and top would take probably two minutes each setting. The actual quantity of products obtained would thus be

diminished to 910 products per hour, on the average. Still at this rate a complete set of D numbers for Two Joint Lives of Males or Females, or these combined, could be written down in less than six hours.

Eighth Annual Report of the Superintendent of the Insurance Department—State of New York.

ALTHOUGH the general principles upon which the business of Insurance is based, must be the same wherever it is carried on; yet its particular developments must of necessity vary in countries under different laws, and with a public opinion differently educated. An investigation therefore of the book before us, with the view of pointing out those points wherein the practice of our Trans-Atlantic Brethren differs from ours, may not be without advantage to the interests of Insurance in this country.

The first point which attracts our attention, is the public advantage which results in the United States, from the publication in one volume by an accredited State Officer of the accounts of all the different Insurance Companies doing business in the State. In this way is obtained a clear comparative view of the working of each, and the improvements of some Offices and the shortcomings of others are authoritatively submitted to the public gaze; and thus no doubt a great check is given to the founding of bubble Companies, and to the continued existence of those reared upon an insufficient basis.

The appointment of a Government Inspector with power to order Insurance Companies to publish Balance Sheets, may appear at first sight an undue infringement of the liberty of the subject, and an interference with the freedom of trade; but we think it will appear upon investigation that such an appointment in this country with suitable limitations would be productive of much good, and tend greatly to the protection of the public. There might, it is true, be instances where the compulsory publication of accounts would injure and perhaps destroy companies recently founded, which otherwise might have struggled on for years with a greater or less degree of vitality; but we think that the good effects would immensely preponderate. We are of opinion, however, that the functions of such an Officer should be strictly limited to obtaining and making public a true statement of the business transactions and the financial position of each Company.

The Report now under review treats Insurance under four heads—1st Fire, 2nd Marine, 3rd Life, and 4thly Casualty.

Under the head of Fire Insurance the Superintendent remarks that "It is undoubtedly true that of late years the causes of spon-"taneous combustion have increased, and that the multitude of "modern business transactions, necessitating frequent changes "and confined storage of goods, has operated to increase the "destruction of property. Doubtless, also, we are paying an un-"avoidable part of our patriotic Debt in the demoralization which "necessarily results, more or less, from every great and sanguinary "war. It is also stated that our modern methods and plans of "building gravely increase the hazards of fire. The losses are "becoming so alarming that it has been suggested whether the "total abolition of all Fire Insurance might not prove to be a "national blessing." Further on, he says "The average loss for "seven years on Fire risks written, has been ·4472 per cent, and "the average percentage of losses to premiums, 62·51 per cent. "The oscillations of loss however from this average, in different "years, are fearful, and when the much heavier oscillations of "individual companies in various years are taken into considera-"tion, all satisfactory conclusions as to the future 'mortality' of "insured property are demonstrated to be unscientific, delusive "and uncertain. Hence the Superintendent again begs leave to "urge upon all companies, however apparently strong and impreg-"nable, the immediate establishment of an extraordinary Reserve "Fund, in addition to the usual re-insurance fund, for the purpose "of meeting those contingencies of the business which are now "seemingly so certain as to be almost considered ordinary in the "lapse of a single decennium."

It appears that no new Joint-Stock Fire Insurance Company was organised in the state of New York in 1866, that only two Fire Insurance Companies have increased their Capitals in that period and that after formal "special examinations"—"Requisitions or Calls" have been made upon the Stockholders of 6 Companies whose capitals have been impaired by heavy losses since the last Report. The Superintendent states that "The statute in such "cases does not give any right of action in favor of the Company "against the stockholder personally to collect the amount of "deficiency on his stock; the stockholder can be compelled only to "surrender the deficient stock and take out new scrip or a stock "certificate for his proportion of the remaining Capital; the "deficiency can thus be thrown back upon the corporation, and is

" represented by new stock for sale at par, and which when thus
" sold will repair the capital stock to an actual par value, besides
" the good will of the business not included of course in the balance
" sheet of Assets and Liabilities. But if through a general depres-
" sion in the value of Fire Insurance stocks or for other causes,
" this new stock cannot be sold at par, either the corporation must
" be reported to the Attorney-General for dissolution, or its Capital
" must be decreased to accord with its actual Assets. . . . . . The
" Superintendent therefore reluctantly recommended the passage
" of a general statute allowing a reduction of the capitals of Fire
" Insurance Companies, in certain cases, not below the minimum
" fixed for the organization of new corporations."

This appears to us a most proper regulation; and one that might with advantage be adopted in this country. Is it not, indeed, a fraud for a Company which has lost a large proportion of its paid-up capital to hold itself out to the public as possessing a paid-up capital of the original amount? This being admitted, it cannot be questioned that one of the most proper functions of every government is the detection and punishment of frauds of all kinds.

Then follow particulars of the different ways in which the six Companies propose to meet the Requisitions. We have next a list of 33 Companies whose capitals are more or less impaired; of 10 whose capitals—impaired at the date of the last Report—have been since "repaired," and of 71 Companies which have their capitals intact, and exhibit surpluses of various magnitudes. It appears that six Companies have discontinued business during the year, and "are closing up their affairs." A number of tables are then given exhibiting the comparative standing of the 108 Companies doing business according to (1) the amount of paid up capital, (2) the amount of the assets, (3) the amount of the annual "net cash premiums," (4) the percentage of assets to risks, (5) the percentage of losses to net cash premiums. Under the last head, we note that one Company exhibits losses amounting to 232 per cent of the premiums; 11 Companies have losses exceeding 100 per cent of the premiums, and the average of the whole losses to premiums is 76 per cent.

With regard to the Mutual Fire Insurance Companies, it appears that only nine of these Corporations were remaining in business on the 31st December, 1866, and that "The average loss
" of the Mutual companies on the cash premiums received, is not
" much more than half that incurred by Stock companies during

"the year; a result owing undoubtedly to the safer character of "the risks, which are generally on dwellings and farm property, "and also to the personal supervision which the officers are able "to give to their risks."

The Superintendent then gives a list of the Fire Insurance Companies of other States which have been authorized to transact business in the State of New York during the year 1867, and to whom the usual renewal or other certificates of authority have been issued to their New York State Agencies. This Statement further shows the hold which the Government has over these Companies, as in the event of the Capital of a Company being impaired beyond a certain extent, the Certificate is either at once revoked or not renewed at the expiration of the year.

The Capitals of three Fire Insurance Companies of other States have been increased during the year 1866; the addition to the capital of one was made from accumulations, the two others made their additions by new subscriptions to Capital Stock, which were paid up in Cash.

Upon a complete review of this branch of the subject, the Superintendent concludes with the following encouraging remark. "The New York State Companies have more than trebled their "premiums in five years (1862–1866). This fact, although "coupled with another that the losses have increased in a still "larger ratio, indicates that by a more careful and thorough super- "vision of risks, and the adoption of other reforms, future success "and prosperity are attainable."

With reference to Marine Insurance, which is much more of an uncertain character than the others—the Superintendent remarks that "Both our Fire and Marine losses now verge upon national "calamities, requiring the deepest thought of the political econo- "mist and statesman. The present fearful percentage of loss is "too excessive, and must in some manner be reduced and not "merely covered by Insurance. It is often said that our Marine "Insurance interests scarcely affect any one except a few mer- "chants and individuals in the city of New York; primarily it "may be so, but ultimately the millions paid for Marine premiums "are indirectly taxed upon the whole body of the people of this "State, and to a considerable extent on the whole country, reaching "in some form almost every citizen of the Republic." He considers that the amount of Capital at present required for the organization of a Marine Insurance Company is entirely too small, and recommends that "at least $500,000 capital should be required

"for the incorporation of a Marine Insurance Company in any "part of the State."

An attempt has been made to renew the old method of Insurance by means of individual underwriting as follows—"An "unincorporated voluntary Association of about one hundred "individual underwriters has been recently organized in the city "of New York under the title of the 'United States Lloyds,' for "the purpose of writing Marine risks to an amount not exceeding "one thousand dollars for each member or firm on any one risk. "The plan is for each member to pay in and maintain intact in "the hands of an Advisory Committee the sum of one thousand "dollars to meet losses, and to authorize Messrs Robinson & Cox, "as their Attorneys, to underwrite risks and fix rates of premium "in their discretion under the general advice and direction of the "Advisory Committee." But the Superintendent is not informed as to the practical working and results of this attempt.

We come now to the very important part of the Report referring to Life Insurance.

It appears that "The year 1866 has been the most fruitful "year ever known in this State for the organization of Life "Insurance Companies. Six Charters have been filed and five "companies fully organized and incorporated, some of which have "met with marked and unusual success."

The following remarks appear to us to possess special interest, as going even beyond what has yet been attempted in this country. "Some of these companies have adopted new and peculiar modes "of transacting business and attracting patronage, one (the "Atlantic) reducing the rate of premium ten per cent to those who "in cases of sickness are habitually treated upon the principles of "the Homœopathic, as contra-distinguished from the Allopathic "school of Medicine. Another Company the (American Popular) "takes a wider hygienic range, and from all the physical and "moral signs of longevity exhibited by an applicant, and the special "law of family vitality, as deduced from ancestral tendencies, both "in the direct and collateral relatives, essays to modify the general "law of average expectation of human life as stated in the Table of "Mortality, so as to accord with the special law governing the "individual case, rating his expectation of life or assumed age up "or down the scale of Table expectation, according to the particular "quantum of his unexpended vital force." The experience of this country gives no encouragement to such novelties as these; for whenever anything of the kind has been attempted, it has been invariably abandoned after a few years' trial.

An act lately passed by the New York Legislature, permitting under certain conditions the "registration" of Policies appears deserving of consideration. The object and result will be best seen from the Superintendent's own words. "The Registry "system combines the advantages of individual and corporate "enterprise with governmental custody, supervision and guardian-"ship of funds. The practice, however, not being compulsory on "either policyholders or companies, must succeed if at all on its "own intrinsic merits. In many localities not familiar with the "status and standing of companies or of their officers, parties can "sometimes effectuate their purposes more satisfactorily by the "registration of their policies, thereby compelling a company to "deposit, in addition to its general deposit of $100,000 made by "all companies, a further special amount equal at all times "to the net present value or re-insurance fund of such policies." And again, "It will be noticed that the above Act is silent "upon the subject of State liability to the policyholder. It "is permissive, not imperative in its character, allowing any Life "insurance company, duly authorized to make insurance on lives "in this State, to deposit certain securities in the Insurance "Department, to be held by the Superintendent, in trust, until "the obligations of the depositing company, under its registered "policies, shall be fully liquidated, cancelled or annulled. In this "manner the State, through the Insurance department, becomes "the custodian of the re-insurance fund of registered Life policies. "The Superintendent does not understand that any technical or "legal liability on the part of the State is created thereby, except "faithfully and with ordinary care and diligence to perform the "duties incident to its trust relations." The Superintendent further goes on to say that "An amendment to the general "Registry Act for Life Insurance policies, limiting and restricting "the securities to be deposited solely to Registered New York "State stocks, would entirely eliminate all risk of loss on the part "of the State, appreciate the market value of our stocks, and "return them practically funded to our own vaults. The State "could then, without any risk, become responsible for, or even "guarantee the payment of registered Life policies and annuities. "Under the present system of registration, although the State is "not legally responsible for the payment of the registered policies, "yet there is a moral responsibility attached to the trust which "might, under certain circumstances, compel the Legislature to "assume the payment of policies. The Superintendent, therefore,

"respectfully recommends to your Honourable Body, an amend-"ment of the Registry Act, limiting the deposits to New York "State stocks, and then assuming the same State liability for "registered policies in case of their non-payment by the company, "as that contained in the Banking Act above cited, in reference to "the payment or redemption of bank bills or circulating notes. "Such a system well guarded and regulated would make a Life "policy or annuity just as safe as the stocks of the State of New "York, with no real liability on the part of the State except to pay "its own State debt."

The following remarks are important as showing how much more successfully the science of legislation is studied in New York than in this country. "Several applications having been made to "the present Legislature by other companies for the passage of "special Acts similar to that passed for the North America in "1866 (*Chap.* 576), and one Act having been passed this session "for the Atlantic Mutual Life Insurance Company (*Chap.* 445, "*Laws of* 1867), it was deemed expedient to abolish all special "laws of this nature, and in accordance with our usual State policy "to pass a general Act allowing all companies to register policies "at their option."

The Life Offices doing business in the State of New York are now required to furnish very full and detailed statements of their Liabilities, Income, and Expenditure. In reference to this the Superintendent remarks, "The business of Life Insurance was "assuming such enormous proportions and national importance, "that it was deemed necessary and conducive as well to its "own preservation as to the public welfare that the companies "should be subjected to more detailed analytic and exhaustive "exhibits of their Assets and Liabilities, Income and Expendi-"tures, and a more perfect *exposé* of their various modes of opera-"tion and that stricter surveillance which always accompanies "publicity. Sunlight is not more conducive to healthy vegetable "life than Publicity to corporate well-being. Probably no "corporations in any country were ever subjected to such a "complete disintegration of their internal mechanism and to so "minute an inventory of Resources and Liabilities; but no "corporations in any country were ever so trusted with the public "confidence or ever reaped such princely and progressive Incomes, "with liabilities payable mostly to the next generation."

The magnitude of the American Life Offices far surpasses anything that this country can show. It appears thus to be a law of

nature that everything in the New World, whether in the realm of nature, or art, shall be on a larger scale than in the Old World. Mr. Barnes after remarking that the income from new premiums of two Companies only in this country—the Scottish Widows' and the Gresham—exceeds £40,000, adds that there are American Companies engaged in Life assurance business exclusively, whose income from *new* premiums is five-fold that of the "Scottish Widows' Fund" and "Gresham" combined. It seems doubtful whether the assured derive any advantage from the magnitude of the Offices; for the expenses of management appear to bear as large a proportion to the premiums as in England; if, indeed, the proportion be not larger.

A series of very interesting tables are given in the Report showing the progress during the last eight years of all the Life Offices doing business in the State; and it is impossible in this country to obtain statements at all approaching these in accuracy or fulness. It appears that on 31st Dec. 1866 there were 39 Offices only doing business, which had 305,390 policies in existence, insuring £173,021,175; the annual premiums amounting to £7,239,520; that the amount of the losses in 1866 was £1,284,734; and that the total amount of the assets was £18,317,406—the average amount of each policy being £566. On these figures we would remark that the average premium is much higher than in this country, and that the losses are very light—probably because of the recent selection of the majority of the lives assured.

It appears that in some Companies a portion of the assets consists of "premium notes," or, as we call them, "half credit premiums." The Superintendent thinks that in some instances the proportion these bear to the total assets is excessive, and he desires some legislation to cure the evil. He remarks "The usual practice of attempting to hold only enough assets in promissory notes to offset current dividends, has been flagrantly violated, and the pressure and competition for business is so sharp and reckless that the tendency to accumulate an excessive amount of premium note assets appears to be increasing with some Companies."

A still more objectionable practice prevails in some instances of reckoning among the assets the "commuted commissions" paid on existing policies.

We learn that it is proposed to establish an "American Chamber of Life Insurance"—an association of the officers and

actuaries of the various Life Insurance Companies in the United States "for consultation on the common interests of the business, "and for its general advancement and improvement in this "country, and also for initiating and maintaining intercourse with "learned foreign bodies of this nature." Mr. Barnes does not consider however that the era "has yet been reached which can "sustain a purely scientific association of Actuaries and mathema- "ticians like the Institute of Actuaries"—to which he pays the following handsome tribute. "The American Life Insurance "interests, in common with those of the whole world, are greatly "indebted to the thorough and scientific labors of the Institutes "of Actuaries, both in London and Edinburgh; and no incon- "siderable portion of our own success has been the legitimate "fruit of their discussions and researches."

At the present time, when the "Life Policies Nomination Bill" is before Parliament, the following remarks of Mr. Barnes will, no doubt be read with interest. "The well being of the State and "its general prosperity would be promoted by a Constitutional "provision declaring that a whole life policy of insurance, with "equal annual premiums during life, for a sum not exceeding "ten thousand dollars, payable at death to the wife and children "of the insured, or any or either of them may, if so originally "declared in the policy, become and shall be exempt from the "claims of the husband's creditors."

Casualty Assurance Companies do not appear to flourish in the State of New York; for of the 4 already organized, one is preparing to surrender its present Charter; another has decided to discontinue business; the third has resolved to wind up its affairs as soon as possible; and the fourth has applied to the legislature for the privilege of engaging also in the business of Life Assurance. It appears however that another Company has been projected with a fair prospect of organisation.

HOME AND FOREIGN INTELLIGENCE.

THE STANDARD LIFE ASSURANCE COMPANY.

Established 1825.

INVESTIGATION REPORT AS AT 15TH NOVEMBER, 1865.

EXTRACTS FROM THE REPORTS.

PROGRESS OF THE BUSINESS.

The period from 15th November 1860 to 15th November 1865, to which this Report bears reference, has been marked by very great success in the progress of the Company's business, and the Directors have much pleasure in submitting to the Special Meeting now assembled an account of its operations during that period, with the result of the investigation into its monetary position at the date of the Quinquennial Balance.

Before communicating the results of the investigation, it will be well to recapitulate the results communicated to the Annual General Meetings during the period under review:—

In the year 1861 the New Assurances effected amounted to	£503,854	18	0
In " 1862 " "	506,120	0	0
In " 1863 " "	643,960	0	0
In " 1864 " "	805,980	6	6
In " 1865 " " (inclusive of New Assurances effected by the Colonial Company during 1865 for behoof of the Standard Company. amounting to £400,255 : 19s.)	1,374,450	1	0
In all . .	£3,834,365	5	6

Again, if a comparison be instituted between each Quinquennial Period during the last twenty years, the result stands as follows:—

From 15th Nov. 1845 to 15th Nov. 1850, New Assurances	£2,146,641	12	9
From 15th Nov. 1850 to 15th Nov. 1855, New Assurances	£2,492,988	6	7
From 15th Nov. 1855 to 15th Nov. 1860, New Assurances	£2,815,455	3	0
From 15th Nov. 1860 to 15th Nov. 1865, New Assurances	£3,834,365	5	6

THE REVENUE of the Standard Company was in 1845 .	£103,371	8	5
" " " " " 1850 .	£169,151	16	4
" " " " " 1855 .	£237,480	1	9
" " " " " 1860 .	£304,161	13	7
" " " " " 1865 .	£661,195	0	0

The Funds at the date of the Valuation amounted to £3,651,683.

[We find it also stated in the course of the proceedings that the total sum assured is £15,710,982; that the average amount insured on each policy is £571; that the average age of the assured at entry is a little over 36; and that the average rate of interest on the investments is nearly 4½ per cent. We do not however find any statement of the amount of the divisible surplus.]

PRINCIPLES OF VALUATION.

The tables and data used are the same as were employed in 1860, and at the four previous investigations.

The safety of the mode of procedure in an investigation arises from employing a Table of Mortality which gives a death-rate in excess of the mortality among assured lives, proper care being used in selection, and from the adoption of a low rate of interest, such as the Company can confidently calculate on realising at all periods, with the prospect of a margin for safety and profit. Carlisle 3 per cent Tables, which form the chief basis of the Company's calculations, answer all these requirements. Of equal importance is the question of what is technically termed "loading" —that is, the percentage added to the premiums above the rate necessary to secure the capital assured at death. Out of the loading comes the annual fund for expenses and profit, being in addition to the profit arising on investments and the selection of lives. This matter is a point of great importance, not less so than the question of basis or data, and on its integrity depends greatly the safe and thorough investigation of the affairs of an Assurance Company. The reserve so made is fortunately not an arbitrary amount, as the whole loading must be set aside, not this portion or that portion of it; and the Directors have pleasure in stating that no less a sum than £91,530 per annum of this Company's premium income (worth, probably, fifteen years' purchase) has been reserved as an unvalued asset—that is, no value has been placed upon it, although it is a reliable source of income in future years. This reserve is derived from the premium income of the Company from all sources.

TABLE SHOWING THE BONUS ADDITIONS TO POLICIES

Declared from the Profits of the Company realised during the Five Years ended 15th November 1865.

Date of Policy prior to 15th Nov.	Sum in Policy.	Bonus Additions Declared.		Total Bonus Additions.	Sum in Policy, with Bonus Additions.
		Previously to 1865.	In 1865.		
1825	£1000	£1440 0 0	**£287 0 0**	£1727 0 0	£2727 0 0
1830	1000	1115 0 0	**252 0 0**	1367 0 0	2367 0 0
1835	1000	790 0 0	**217 0 0**	1007 0 0	2007 0 0
1840	1000	515 0 0	**182 0 0**	697 0 0	1697 0 0
1845	1000	302 10 0	**147 0 0**	449 10 0	1449 10 0
1850	1000	152 10 0	**112 0 0**	264 10 0	1264 10 0
1855	1000	57 0 0	**77 0 0**	134 0 0	1134 0 0
1860	1000	8 0 0	**42 0 0**	50 0 0	1050 0 0
1865	1000	..	**7 0 0**	7 0 0	1007 0 0

Bonus Additions do not vest until the Policies have been in existence for Five Years from the date of the risk commencing.

THE ECONOMIC LIFE ASSURANCE SOCIETY.

Established 1823.

Eighth Quinquennial Report.

The Directors of the Economic Life Assurance Society feel much pleasure in presenting to its Members their report of the result of the investigation made into the affairs of the Society at the close of the quinquennial period which terminated on the 31st December, 1863.

Since the last Division, 2,641 Policies, assuring £2,050,788, have been issued, giving an annual average of 528 Policies of £777 each—a large and steady increase of the business.

The sum of £72,702 has been received during the five years in new Premiums, being at the rate of £14,540 a year.

The total income from Premiums, which in 1859 was £182,429, now amounts to £214,104, indicating an average annual increase of £4,385, after allowing for loss of income from discontinued Policies; while the gross income from all sources has increased at the rate of £10,230 per annum.

Claims have arisen during the five years on 794 Policies assuring £624,327, and carrying Bonuses to the amount of £116,899.

In addition to the Bonuses on Policies upon which claims have arisen, the sum of £87,149 has been paid as bonus in other ways, such as in reduction of bonus liability by cash payment, reduction of premium, purchase, &c., making a total of £204,048.

In the valuation of the Assets, an ample margin has been allowed for possible fluctuation of the Funds; and in the valuation of the Liabilities, the *risk* Premiums only have been taken into account. The remaining portion of the Premium income, after defraying the expense of management, commission, &c., will accumulate till the next Division, when the amount realised will be found in the surplus. By this arrangement, old and new Assurers contribute rateably to the expenses of management, and no profit is declared by anticipation.

The Assets, consisting of Funded Property, Mortgages, Life Interests, and Reversions, Premiums due on 31st December (since paid), Interest accrued on Investments, Balance at Bankers and in hand, amount to £2,315,129. 19*s.* 2*d.*

The Liabilities, consisting of the values of Policies and the Bonuses already declared, claims accrued in 1863 but due in 1864, commission, taxes, and sundry small accounts, amount to £1,964,739. 1*s.* 7*d.* There is therefore, after making provision for every known liability, a surplus of £350,390. 17*s.* 7*d.*

The Directors recommend that £329,890 of this surplus be distributed as absolute Bonus; and that the remaining sum of £20,500. 17*s.* 7*d.* be retained for the payment of annual, contingent, and conditional Bonuses.

It is further recommended, that out of this sum of £20,500. 17*s.* 7*d.* an annual Contingent Bonus of £1 per cent. per annum be added to the absolute Bonus on such Policies, now entitled to participate, as shall become claims during the current quinquennial period, viz.:—

On Policies which become claims in		1864,	£1	per cent. on sum assured.	
"	"	1865,	£2	"	"
"	"	1866,	£3	"	"
"	"	1867,	£4	"	"
"	"	1868,	£5	"	"

To those Policies which are not entitled to participate in the present Bonus, by reason of five annual Premiums not having been paid upon them, but on which claims may arise after the payment of the fifth annual Premium and before the next quinquennial investigation, they propose to add a Bonus of like amount as if five annual Premiums had been paid prior to the present Division.

The sum of £329,890 will produce reversionary Bonuses amounting to £506,300, yielding a percentage ranging from 5 to 34, or $9\frac{1}{8}$ on the average of the sums assured; and a percentage ranging from 26 to 160, or $59\frac{1}{2}$ on the average, on the Premiums received in respect of which the Bonus is allotted.

The same options are offered as at the last Division, and to the present Bonus may be taken either

1st. In Money;
2nd. In an addition to the sum assured;
3rd. In a reduction of the Premiums for five years only; or
4th. In a reduction of the Premiums for the remainder of life.

The Society now assures by 9,022 Policies the sum of £7,233,564, and has an Assurance Fund amounting to £2,272,385. 11*s*., and an Annual Income of £307,475. The large number of assurances in force, affords a protection to the Society against those deviations from the average which attend a paucity of numbers.

EQUITY AND LAW LIFE ASSURANCE SOCIETY.

Established 1844.

Bonus Report, 1864.

The Fourth quinquennial period of the Society's operations having closed on the 31st December last, the Directors have caused a careful valuation to be made by the Actuary of the assets and liabilities of the Society as at that date; and have now in conformity with the provisions of the Deed of Settlement, to report the results to the Proprietors and the Assured.

It will be convenient in the first instance to give a summary of the progress of the Society since the last valuation. In the five years in question, there have been issued 805 new policies insuring £1,159,619 the average amount of each policy being £1,440. In the previous five years, the number of policies issued was 725, insuring £792,485 and averaging £1,093 each.

It is worthy of note that during the last five years the practice of effecting insurances against the birth of issue, in connection with loans on contingent reversionary interests, has grown into importance. Up to the present date, such insurances have been effected with this Society to the

extent of £98,320; and the premiums received in respect of them, have amounted to £5,869.

On the 31st December, 1859, there were in force 1,336 policies, insuring £1,403,880; and adding to these the policies since issued, there are 2,141 policies insuring £2,563,499 to be accounted for. Of these, 105 insuring £72,825, have become claims; 280 insuring £311,908, have terminated by lapse, surrender, or expiry, leaving 1,756 policies in force on 31st December last, insuring £2,178,766.

The number and amount of the policies in each class of Assurance are shown in the following table, in which the non-participating policies are distinguished from the participating:—

ABLE SHOWING THE NUMBER AND AMOUNT OF THE POLICIES IN FORCE ON 31ST DECEMBER, 1864.

Class of Assurance.	PARTICIPATING POLICIES.				NON-PARTICIPATING POLICIES.		
	No. of Policies.	Sums Assured.	Existing Bonuses.	Annual Premiums.	No. of Policies.	Sums Assured.	Annual Premiums.
hole Life..........	1,241	£1,439,774	£67,798·3	£45,285·074	321	£433,693	£15,382·924
imited Payments	8	14,200	329·0	778·675	1	3,000	195·000
scending Scale	1	5,000	..	68·750	9	13,500	313·368
ndowment Assurances	8	3,650	87·0	175·238	5	10,750	406·646
int Lives	10	12,300	706·5	784·196	4	1,650	125·280
ast Survivor	8	16,800	1,133·5	300·067	7	13,900	275·350
ntingent	..	..	..	..	63	89,590	1,387·565
					410	£566,083	£18,086·133
ndowments	..	..	..	..	2	200	..
erm Policies	..	..	..	..	35	54,450	1,219·945
ssurances against Issue	..	..	..	..	22	66,309	..
xtra Premiums	..	..	..	..	..	..	585·917
mmediate Annuities ..	..	..	..	..	5	(923·333 per annum)	..
eferred ditto ..	..	..	..	..	1	(30·000 „)	9·525
eversionary ditto ..	..	..	..	..	5	(745·000 „)	138·858
TOTAL........	1,276	£1,491,724	£70,054·3	£47,392·000	480	£687,042 .. (&£1,698·333 ℘ ann.)	£20,040·378

	No. of Policies.	Sums Assured.	Existing Bonuses.	Annual Premiums.
tal Participating and Non-Participating Policies	1,756	£2,178,766	£70,054·3	£67,432·378

It will be noticed that the non-participating policies amount to about 30 per cent. of the whole business; and the profit arising therefrom far exceeds the proportion—one tenth of the whole—which is appropriated, under the provisions of the Deed of Settlement, to the Proprietors. Thus, the Assured who participate in the profits, divide among themselves more than the whole of the profits derived from their own policies.

In estimating the liability of the Society under its various insurance contracts, it has been the wish of the Directors to strengthen the position of the Society by making an ample reserve, rather than to divide the largest sum which circumstances might seem to justify. A very large profit has been derived during the last five years from the claims being

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much lighter than could have been possibly expected, the losses having reached only 55 per cent. of the anticipated amount; but the Directors consider it would be unwise to divide the whole of this profit on the present occasion. The process of valuation employed has therefore been of the most stringent character.

The Table of Mortality made use of has been that known as the "Experience Table," which would appear to be the most suitable, as having been derived from observations on assured life, furnished by seventeen Insurance Companies. The reserve obtained by the use of this Table is considerably larger than that given by any of the other Tables commonly employed. The rate of interest assumed in the calculations is three per cent., being the rate commonly adopted for the purpose, as the highest which can with prudence be assumed as likely to prevail permanently during the currency of the policies. The whole of the loading, or addition to the net premium for expenses, contingencies, &c., has been thrown off in estimating the value of the future premiums. In these and other respects, the greatest care has been taken to avoid everything in the nature of anticipation of profits not yet realized.

With these explanations the Directors would call attention to the following Balance Sheet in which the position of the Society on 31st December, 1864, is clearly set forth.

BALANCE SHEET, 31ST DECEMBER, 1864.

LIABILITIES.

	£	s.	d.
Value of £1,491,724 assured under 1,276 Policies with profits . .	777,364	14	0
Value of £70,054 Bonuses thereon	40,926	12	0
Value of £566,083 assured under 410 Policies without profits . .	271,145	0	0
Reserve for Short Term Insurances, Extra Risks, Special Cases, &c. .	13,219	8	0
Claims announced and other liabilities	919	4	0
Balance,—being the excess of Assets over Liabilities . . .	72,357	17	7
	£1,175,932	15	7

ASSETS.

	£	s.	d.	£	s.	d.
Amount of Assurance Fund as per printed account .				383,966	5	7
Value of £47,392 Annual Premiums on policies with profits	700,825	2	0			
Less reserve for expenses, future bonuses, and contingencies	140,020	14	0			
				560,804	8	0
Value of £18,086 Annual Premiums on policies without profits	226,352	18	0			
Less reserve for expenses, &c.	29,744	10	0			
				196,608	8	0
Value of Reassurances for £255,348				34,553	14	0
				£1,175,932	15	7

The Directors recommend that of the above Balance of £72,357 a sum of £2,400 should be appropriated to reduce the price at which the Society's house stands in the books; and that the remaining sum of £69,957 be actually divided. The share of the Proprietors will be £6,995 14*s.*, which will allow of the payment of an increased dividend for the ensuing five years

at the rate of 8*s.* 6*d.* per share, or $8\frac{1}{2}$ per cent. on the amount originally paid. The amount to be divided among the Assured will be £62,961 6*s.* and the amount of the policies which will participate on the present occasion, being effected on the participating scale and of more than one year's standing, is £1,339,608. At the last Division of Profits the sum of £39,500 was divided among policies for the sum of £925,306. If the same relation still subsisted, the sum to be divided among the Assured would be £57,186. The sum now to be divided is, therefore, considerably larger in proportion; and this, notwithstanding that a larger proportionate reserve has been made.

In distributing the above sum among the Assured, care has been taken to adjust equitably the shares of persons insuring at differing periods in the Society's existence. A somewhat larger bonus will be given to the persons who insured many years ago, than to persons who have insured at the same age more recently; but this difference is proportioned to the larger profit derived in the former case; and *no advantage is given to the older Assured at the expense of the more recent.* A larger bonus will be also given to those persons who chose the reversionary bonus at former divisions, than to those who have received the value of the former bonuses in cash, or Reduction of Premium.

The principle on which the distribution has been made, will be better understood when it is stated, that the average rate of interest at which the Funds of the Society (including the unproductive assets) have been improved during the last five years, has been £4 8*s.* per cent. per annum, after deduction of Income-tax. In all the valuations, it has been assumed that three per cent. only would be realized; and the profit from this source upon the amount of funds on 31st December, 1859, forms a considerable sum, of which persons who have insured subsequently, have contributed no part.

The general results of the four Divisions of Profit are shown in the following Table of the total additions made to 31st December, 1864, to policies of £1000 each:—

Age at Entry.	NUMBER OF PREMIUMS PAID.			
	Twenty.	Fifteen.	Ten.	Five.
20	£303 10	£228 0	£161 10	£71 10
30	342 10	250 10	176 0	77 0
40	385 0	280 10	197 0	85 0
45	411 10	303 10	210 0	90 10
50	438 0	334 0	231 0	99 10
55	..	382 10	265 0	113 10
60	..	450 10	316 10	133 10

NOTES AND QUERIES.

SOLUTIONS OF THE SECOND YEAR'S EXAMINATION QUESTIONS.

Mr. Marcus N. Adler points out that the answer given on p. 147 to Question (4) in the paper of 1864 is incorrect; and that the values of the expectations of A and B are really $\frac{16}{31}$ and $\frac{15}{31}$. His demonstration is as follows:—

Let A and B's respective chances be represented by A and B. Let p be the chance of winning of the player about to commence. Then $A=p$; and B or B's chance of winning depends on two independent events, (1) that he becomes the first player, the probability of which is $\left(1-\frac{1}{2^4}\right)$; and (2) that he wins, which is p; and since one must win,

$$\therefore p+\left(1-\frac{1}{2^4}\right)p=1.$$

$$\therefore A=\frac{1}{2-\frac{1}{2^4}}=\frac{16}{31}: \qquad B=\frac{1-\frac{1}{2^4}}{2-\frac{1}{2^4}}=\frac{15}{31}.$$

CORRESPONDENCE.

ON THE ADJUSTMENT OF PREMIUMS FOR LIFE ASSURANCE IN REFERENCE TO EXTRA RISKS.

To the Editor.

Sir,—In my former letter I have examined most of the cases of Assurance on single lives likely to present themselves in practice, and I have shewn how far the ordinary method of determining the Extra Premium in such cases agrees with the supposition of a constant extra risk, upon which that method is professedly based. The truth or error of the supposition is really immaterial, the point at issue being one of consistency only,—but as the importance to be attached to the subject will doubtless be much enhanced if it should appear that, as regards climatic influences, the hypothesis of a constant or uniform extra risk is actually borne out by such facts as are within our reach, I proceed now to shew that such is in reality the case.

Indeed the striking similarity in the progression of the series contained in the last two columns of the Table given at p. 164 is sufficient to beget a suspicion of the fact asserted. The two columns in question comprise the results of a comparison of the *hypothetical* with the *actual* annuity values for five equidistant ages, from 20 to 60 both inclusive. The following

Table, however, which gives the mortality for each interval, must be considered much stronger evidence,—each of the six terms of which it consists forming a distinct and entirely independent testimony to the truth of the hypothesis. It contains the average "rate" and also the corresponding "force" of mortality in the several indicated intervals of age, first by Mr. Neison's Bengal Military Table, and secondly by the Carlisle Table. The difference between these, which is given in the next column, is the *extra* force which by the hypothesis is supposed to be constant for all ages. The result, I hold, accords with the hypothesis fully within the limits of the error which we know, by ample experience, attaches even to the best and most trustworthy Tables.

Ages.	Average Rate of Mortality.		Corresponding Force of Mortality.		Differences of Columns (3) and (4).
	(1) By Carlisle Table.	(2) By Neison's Bengal M. Table.	(3) By Carlisle Table.	(4) By Neison's Bengal M. Table.	
21–24	·00702	·02307	·00704	·02334	·01630
25–32	·00893	·02534	·00897	·02567	·01670
33–40	·01099	·02860	·01105	·02902	·01797
41–48	·01446	·03140	·01457	·03190	·01733
49–56	·01582	·03318	·01595	·03374	·01779
57–64	·03226	·04787	·03279	·04904	·01625
Sum of each Column	·08948	·18946	·09037	·19271	·10234
Average of do.	·01491	·03158	·01506	·03212	·01706

The first interval comprises four years only, the remainder eight each, so that the last interval terminates with the age 64, with which Mr. Neison's data are exhausted. The "force" of mortality is deduced from the corresponding "rate" (μ) by the formula $\frac{\mu}{1-\frac{1}{2}\mu}$, which, if I recollect rightly, is due to Dr. Farr. It denotes the force of mortality at the middle, not at the commencement, of the year,—and may be thus demonstrated:

$$\frac{d\,L_{x+\frac{1}{2}}}{dx}=\Delta L_x-\tfrac{1}{24}\Delta^3 L_{x-1}+\ldots\text{(See Vol. 13, p. 353)}$$

$$L_{x+\frac{1}{2}}=(L_{x+\frac{1}{2}})-\frac{1}{2}\cdot\frac{1}{2^2}(\Delta^2 L_{x-\frac{1}{2}})+\frac{1.3}{2.4}\cdot\frac{1}{2^4}(\Delta^4 L_{x-\frac{3}{2}})-\ldots\;*$$

where $(L_{x+\frac{1}{2}})$, $(\Delta^2 L_{x+\frac{1}{2}})$, &c., denote the mean of two successive terms, as $\frac{1}{2}(L_x+L_{x+1})$, $\frac{1}{2}(\Delta^2 L_x+\Delta^2 L_{x+1})$, &c. Hence stopping at the first term of each series we have, $(L_{x+\frac{1}{2}})$ being equal to $L_x+\frac{1}{2}\Delta L_x$,

$$F_{x+\frac{1}{2}}=-\frac{dL_{x+\frac{1}{2}}}{L_{x+\frac{1}{2}}dx}=\frac{-\Delta L_x}{L_x+\frac{1}{2}\Delta L_x}=\frac{-\frac{\Delta L_x}{L_x}}{1+\frac{1}{2}\cdot\frac{\Delta L_x}{L_x}}=\frac{\mu_x}{1-\frac{1}{2}\mu_x}.$$

* I am not aware that this formula has ever been given before. It will be found very convenient for bisecting the interval of two successive terms of a series; and may be easily deduced from the ordinary formula.

Having thus ascertained the law which (to the best of our information) appears to govern the influence of climate upon European life in India, we are justified in inferring that the same law operates in other countries. Indeed the conclusion is inevitable until it shall have been shewn, in any particular instance, that some other law prevails. Our requirements, therefore, for determining the true law of mortality among Europeans residing abroad are limited to the determination of the values of the single constant representing the extra risk for the respective climates. And herein lies the great importance of the discovery of general laws, viz., that by their aid we are enabled to economize our facts, and, by bringing them all to bear upon a single point, compel them, as it were, to yield us information which we should otherwise be unable to obtain. As an instance of this I may refer to the Tables published in Vol. 7, p. 134, of this *Journal*, embodying a considerable collection of facts relating to foreign risks. These facts, although insufficient to serve for the construction of independent Tables of mortality for different climates, are yet numerous enough to enable us to determine, with a tolerable degree of accuracy, the single constant required for the adjustment of the Home Table to each case; and I may perhaps on some future occasion ask the indulgence of your readers for an analysis of the experience referred to.

I am, Sir,
Your very obedient servant,
W. M. MAKEHAM.

10, *King Street, Cheapside,*
27th February, 1868.

DEMONSTRATION OF A FORMULA FOR INTERPOLATION.

To the Editor of the Assurance Magazine.

SIR,—I have been asked on more than one occasion how the formulæ were obtained which I used in graduating the mortality among the males of the peerage (*Assurance Magazine*, vol. xii., p. 221); and as the question may not be without interest to some readers of the *Journal*, I will now state the method.

I had noticed the following resemblance between two sets of expressions. If a series u_0, u_5, u_{10}, &c. is differenced, and $\Delta u_0, \Delta^2 u_0, \ldots$ be the initial terms of each order of differences; $\Delta^4 u_0$ being constant; then, as is shown on page 23 of the current volume of the *Journal*,

$$\delta u_0 = \frac{1}{5}\Delta u_0 - \frac{2}{5^2}\Delta^2 u_0 + \frac{6}{5^3}\Delta^3 u_0 - \frac{21}{5^4}\Delta^4 u_0 \qquad (1)$$

$$\delta^2 u_0 = \frac{1}{5^2}\Delta^2 u_0 - \frac{4}{5^3}\Delta^3 u_0 + \frac{16}{5^4}\Delta^4 u_0 \qquad (2)$$

$$\delta^3 u_0 = \frac{1}{5^3}\Delta^3 u_0 - \frac{6}{5^4}\Delta^4 u_0 \qquad (3)$$

$$\delta^4 u_0 = \frac{1}{5^4}\Delta^4 u_0 \qquad (4)$$

Also, if Σ_1 be the sum of the first five terms $u_0 + \ldots . + u_4$ of a series, Σ_2 the sum of the next five terms u_5 to u_9, and so on; and $\Delta\Sigma_1$, $\Delta^2\Sigma_1$, $\Delta^3\Sigma_1$, the differences of these sums, their third difference being constant instead of the fourth, it may be proved by subtracting from one another the values of Σ_1, Σ_2, &c. in terms of u_0, δu_0, $\delta^2 u_0$, &c. that

$$\delta u_0 = \frac{1}{5^2}\Delta\Sigma_1 - \frac{4}{5^3}\Delta^2\Sigma_1 + \frac{16}{5^4}\Delta^3\Sigma_1 \quad (5)$$

$$\delta^2 u_0 = \frac{1}{5^3}\Delta^2\Sigma_1 - \frac{6}{5^4}\Delta^3\Sigma_1 \quad (6)$$

$$\delta^3 u_0 = \frac{1}{5^4}\Delta^3\Sigma_1 \quad (7)$$

For according to the ordinary formula for the sum of any number of terms of a series,

$$\Sigma_1 = 5u_0 + \frac{5.4}{1.2}\delta u_0 + \frac{5.4.3}{1.2.3}\delta^2 u_0 + \frac{5.4.3.2}{1.2.3.4}\delta^3 u_0$$

$$\Sigma_2 = 10u_0 + \frac{10.9}{1.2}\delta u_0 + \frac{10.9.8}{1.2.3}\delta^2 u_0 + \frac{10.9.8.7}{1.2.3.4}\delta^3 u_0 - \Sigma_1$$

$$\Sigma_3 = 15u_0 + \frac{15 \cdot 14}{1.2}\delta u_0 + \frac{15.14.13}{1.2.3}\delta^2 u_0 + \frac{15.14.13.12}{1.2.3.4}\delta^3 u_0 - (\Sigma_1 + \Sigma_2)$$

$$\Sigma_4 = 20u_0 + \frac{20.19}{1.2}\delta u_0 + \frac{20.19.18}{1.2.3}\delta^2 u_0 + \frac{20.19.18.17}{1.2.3.4}\delta^3 u_0 - (\Sigma_1 + \Sigma_2 + \Sigma_3)$$

$$\begin{aligned} \therefore \Sigma_1 &= 5u_0 + 10\delta u_0 + 10\delta^2 u_0 + 5\delta^3 u_0 \\ \Sigma_2 &= 5u_0 + 35\delta u_0 + 110\delta^2 u_0 + 205\delta^3 u_0 \\ \Sigma_3 &= 5u_0 + 60\delta u_0 + 335\delta^2 u_0 + 1155\delta^3 u_0 \\ \Sigma_4 &= 5u_0 + 85\delta u_0 + 685\delta^2 u_0 + 3480\delta^3 u_0 \end{aligned}$$

If these quantities are differenced, the first column of differences will be

$$\begin{aligned} &25\delta u_0 + 100\delta^2 u_0 + 200\delta^3 u_0 = \Delta\Sigma_1 \quad (8) \\ &25\delta u_0 + 225\delta^2 u_0 + 950\delta^3 u_0 \\ &25\delta u_0 + 350\delta^2 u_0 + 2325\delta^3 u_0 \end{aligned}$$

the second column

$$\begin{aligned} &125\delta^2 u_0 + 750\delta^3 u_0 = \Delta^2\Sigma_1 \quad (9) \\ &125\delta^2 u_0 + 1375\delta^3 u_0 \end{aligned}$$

and the last difference

$$625\delta^3 u_0 = \Delta^3\Sigma_1$$

$$\therefore \delta^3 u_0 = \frac{1}{5^4}\Delta^3\Sigma_1, \text{ as stated in (7).}$$

Substituting this value of $\delta^3 u_0$ in (9) we have

$$\Delta^2\Sigma_1 = 125\delta^2 u_0 + \frac{750}{5^4}\Delta^3\Sigma_1$$

$$\therefore \delta^2 u_0 = \frac{1}{5^3}\Delta^2\Sigma_1 - \frac{6}{5^4}\Delta^3\Sigma_1 \quad (6)$$

and substituting these values of $\delta^2 u_0$ and $\delta^3 u_0$ in (8) we have

$$\Delta\Sigma_1 = 25\delta u_0 + \frac{100}{5^3}\Delta^2\Sigma_1 - \frac{600}{5^4}\Delta^3\Sigma_1 + \frac{200}{5^4}\Delta^3\Sigma_1$$

$$= 25\delta u_0 + \frac{100}{5^3}\Delta^2\Sigma_1 - \frac{400}{5^4}\Delta^3\Sigma_1$$

$$\therefore\ \delta u_0 = \frac{1}{5^2}\Delta\Sigma_1 - \frac{4}{5^3}\Delta^2\Sigma_1 + \frac{16}{5^4}\Delta^3\Sigma_1 \qquad (5)$$

Now it will be seen that the coefficients of $\Delta\Sigma_1$, $\Delta^2\Sigma_1$, &c. in (5) are the same as those of $\Delta^2 u_0$, $\Delta^3 u_0$, &c. in (2); and the same identity of coefficients occurs in (3) and (6) and in (4) and (7). As this identity could not be accidental I proceeded to try substitution of coefficients on a series involving a larger number of differences. The method of obtaining δu_0, &c. from the differences (Δu_0, $\Delta^2 u_0$, &c.) of every hth term of the original series is given by Mr. Neison in his Contributions to Vital Statistics, and by means of it I found the values of $\delta^2 u_0$, $\delta^3 u_0$, &c. in terms of $\Delta^2 u_0$, $\Delta^3 u_0$, &c., putting $h=10$, and supposing the sixth difference constant; the value of δu_0 not being required. These are as follows:—

$$\delta^2 u_0 = \frac{1}{10^2}\Delta^2 u_0 - \frac{9}{10^3}\Delta^3 u_0 + \frac{77{\cdot}25}{10^4}\Delta^4 u_0 - \frac{669{\cdot}75}{10^5}\Delta^5 u_0 + \frac{5895{\cdot}225}{10^6}\Delta^6 u_0$$

$$\delta^3 u_0 = \frac{1}{10^3}\Delta^3 u_0 - \frac{13{\cdot}5}{10^4}\Delta^4 u_0 + \frac{146{\cdot}25}{10^5}\Delta^5 u_0 - \frac{1480{\cdot}5}{10^6}\Delta^6 u_0$$

$$\delta^4 u_0 = \frac{1}{10^4}\Delta^4 u_0 - \frac{18}{10^5}\Delta^5 u_0 + \frac{235{\cdot}5}{10^6}\Delta^6 u_0$$

$$\delta^5 u_0 = \frac{1}{10^5}\Delta^5 u_0 - \frac{22{\cdot}5}{10^6}\Delta^6 u_0$$

$$\delta^6 u_0 = \frac{1}{10^6}\Delta^6 u_0$$

Then reducing the indices of $\delta^2 u_0$, $\delta^3 u_0$, &c., and of $\Delta^2 u_0$, $\Delta^3 u_0$, &c. by unity, and substituting Σ_1 for u_0, these being the changes necessary to convert the formulæ (2), (3) and (4) into (5), (6) and (7), I arrived at the following formulæ, which proved correct in the using.

$$\delta^1 u_0 = {\cdot}01\Delta\Sigma_1 - {\cdot}009\Delta^2\Sigma_1 + {\cdot}007725\Delta^3\Sigma_1 - {\cdot}0066975\Delta^4\Sigma_1 + {\cdot}005895225\Delta^5\Sigma_1$$

$$\delta^2 u_0 = {\cdot}001\Delta^2\Sigma_1 - {\cdot}00135\Delta^3\Sigma_1 + {\cdot}0014625\Delta^4\Sigma_1 - {\cdot}0014805\Delta^5\Sigma_1$$

$$\delta^3 u_0 = {\cdot}0001\Delta^3\Sigma_1 - {\cdot}00018\Delta^4\Sigma_1 + {\cdot}0002355\Delta^5\Sigma_1$$

$$\delta^4 u_0 = {\cdot}00001\Delta^4\Sigma_1 - {\cdot}0000225\Delta^5\Sigma_1$$

$$\delta^5 u_0 = {\cdot}000001\Delta^5\Sigma_1$$

In vol. xii., page 221, however, the u_0 and Σ_1 are omitted: δ stands there for δu_0 and Δ for $\Delta\Sigma_1$.

Yours obediently,

G. W. BERRIDGE.

21, *Fleet Street,*
4th March, 1868.

ON THE RATE OF MORTALITY AT THE PERIOD OF EARLY MANHOOD.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—In an ingenious letter from Mr. Makeham in your last number on the subject of extra risks, that gentleman in discussing the effect of making arbitrary additions to the age of the person whose life is assured, uses the words "Mr. Bailey's theory." Will you allow me a little space, not to gainsay Mr. Makeham's conclusions, with which I am well disposed to agree, but to explain that on the subject of the law of mortality I have no theory to maintain. Some years ago, during the progress of some investigations on this subject, I arrived at the conclusion that Gompertz's theory failed at the period of early manhood. Or to speak more specifically,—that so far from it being uniformly true that the "power to oppose destruction" in the human frame loses equal proportions in equal times, there is abundant evidence to prove that the rate of mortality during the quinquennial period from 20 to 24 years of age is *greater* than in the next succeeding period from 25 to 29.

As it may interest some of your readers to know on what grounds this conclusion was founded, the evidence is here subjoined, not without hopes that its publication may tend to elicit further information on the subject.

Annual Mortality per Cent.

Table.	Age 20 to 24.	Age 25 to 29.
Peerage Males	1·10	·99
Tontine Nominees—Males (A. G. Finlaison)	1·42	1·22
Experience of Life Offices, viz.—		
17 Offices, General, Table F	·89	·76
Eagle, Males	1·16	·78
Economic	·98	·65
Scottish Amicable, Males	·72	·70
London Assurance	·93	·68

In Mr. Fox's paper "On the Vital Statistics of the Society of Friends," *Statistical Journal*, Vol. 22, page 220, the rate of mortality is given for decennial periods of life; and it appears that while the rate from 20 to 30 is ·881, for the following decade it is ·782 only, in that Society.

It seems to me that this evidence is too strong to be impeached; and yet in the graduated tables that have been formed from these observations the peculiarity has been made to disappear.

I am, Sir,
Your obedient servant,

ARTHUR H. BAILEY.

London Assurance,
7, *Royal Exchange*, 9*th March*, 1868.

GERMAN LIFE ASSURANCE INSTITUTE.

To the Editor of the Assurance Magazine.

Dear Sir,—I hope it will not be void of interest to the readers of this *Magazine* to learn, that the example afforded by the Institute of Actuaries has given rise to the establishment of a similar Institution in Germany. By Messrs. R. Busse, Dr. Kanner, Dr. Zillmer at Berlin, Dr. Heym at Leipzic, finanzrath Hopf at Gotha, Dr. Wiegand at Halle, and by myself, the "Collegium für Lebensversicherungswissenschaft" (Institute for the science of Life Assurance) has been founded at Berlin and opened about the beginning of this year. Up to this day it numbers about fifty ordinary Members.

The object of this institution consists according to § 2 of its Statutes in the promotion of life assurance by the cultivation of those technical and scientific studies on which it is founded. According to § 3 the institution will try to reach its aim

1, by papers and debates with reference to the abovenamed topics;
2, by answering questions placed before the Collegium;
3, by the foundation of a library;
4, by publishing reports of its proceedings;
5, by any other means which seem fit to promote its objects.

The members are either ordinary or extraordinary members. Any one, who takes an interest in life assurance may become an extraordinary member by an annual payment of 2 Thalers. In order to become an ordinary member, it is necessary to be proposed by two members, and to prove a qualification either by literary works or in some other way—the contribution of an ordinary member is 6 Thalers annually.

The second meeting was held last week and as proposed by the Council the members resolved that a committee should take the preliminary steps for the formation of a table of Mortality from the data to be collected from the German Life Assurance Offices. It is to be hoped that the Offices will willingly furnish the materials in their possession, thus contributing the means for a most desirable work. This Committee consists of the seven founders of the Institute, of Professor Hülsse at Dresden, and Dr. Laugheinrich and Dr. Bremiker at Berlin.

A German Mortality Experience Table will be of great interest to English Life Assurance offices, as well from a practical point of view, since many English Societies do a good deal of business in Germany, as from a theoretical one in forming means of comparison with the English Experience table already in use, and with the new one now in course of formation.

I am, Dear Sir,
Yours most obediently,

Hamburg, 13 *March* 1868. WILHELM LAZARUS.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

A Comparison of the Values of Policies as found by means of the various Tables of Mortality and the different Methods of Valuation in use among Actuaries. By HENRY WILLIAM MANLY, *of the London and Provincial Law Assurance Society.*

To this Essay was awarded the "Messenger Prize," (1868), the conditions of which will be found further on.

[Read before the Institute, 30th March, 1868.]

LIFE Assurance is an institution possessing such an important influence in the welfare and well-being of Society that it is most essential the principles upon which it is based and conducted should be such as to ensure its lasting stability and prosperity.

After a Life Assurance Office has been once established upon a firm basis, there is no part of its management so important as that relating to the periodical valuation of the liabilities under its policies. It is my object in this essay to inquire into the various methods adopted for that purpose in the present day; and to consider the various results they produce when different tables of mortality and rates of interest are employed in the calculations; and also to ascertain the effects which the employment of different data may have upon the present and future condition of an Office.

The object of the periodical valuations is twofold:—

1st. To determine the exact financial position of the Society at stated epochs; and

2nd. To ascertain if any, and what surplus may be fairly available and disposable as realized profit.

These duties devolve entirely upon the Actuary; and upon his judgment in the selection of the method to be used, the Table whereon the calculations are to be based, and the rate of interest to be employed, depends in a great measure the future prosperity of the Institution.

The Tables of Mortality from which his selection may be made, are very numerous; comprising, amongst others, Tables exhibiting the mortality prevailing amongst whole nations, separate localities and districts, and also among different classes of the population.

The Tables of Mortality generally used in the present day for valuing the liabilities of an Office under its Policies, are — the *Northampton, Carlisle, Davies' Equitable, Combined Experience, Edmonds' Mean Mortality,* and the *English Life Tables.*

Some Offices use Tables formed from combinations of others, which it is impossible to obtain.

The "Northampton Table," once universally adopted by Assurance Offices, was constructed by Dr. Price, and published in his celebrated "Observations on Reversionary Payments." It was originally formed from the Northampton registers of mortality for 36 years (1735 to 1770), but afterwards (4th Ed., 1783) extended to 46 years, from 1735 to 1780.*

The "Carlisle Table" was constructed by Mr. Joshua Milne from the bills of mortality in two parishes in Carlisle for the nine years, 1779–1787, as observed by Dr. Heysham, compared with two enumerations of the population in January, 1780, and December, 1787.†

The "Equitable Table," constructed by Mr. Griffith Davies, was formed from a comparison given by Mr. W. Morgan, in his edition of Dr. Price's works (7th ed., vol. i., p. 182), between the decrements of life in the Northampton Table and those in the Equitable Society, from the year 1768 to the year 1810.‡

The "Combined Experience Table" was the result of an effort on the part of several Actuaries to determine the Law of Mortality which prevailed among assured lives. It contains the combined

* "Observations on Reversionary Payments," by Richard Price, D.D. (7th Ed., edited by W. Morgan). 1812.

† "A Treatise on the Valuation of Annuities and Assurances", by Joshua Milne. 1815.

‡ "Treatise on Annuities, with numerous Tables, &c.," by Griffith Davies, F.R.S.

experience of 17 Offices, and embraces 83,905 Policies; and was printed privately for the Offices in the form of a pamphlet in 1843.*

"Edmonds' Mean Mortality"† is a Table constructed by Mr. T. R. Edmonds, founded upon a theory first propounded by Mr. Gompertz.‡

The "English Life Tables, Nos. 1, 2, and 3," were constructed by Dr. Farr, at the Registrar-General's Office, from the Census Returns and the Registers of Deaths. No. 1§ was formed from the deaths occurring in 1841 compared with the Census of that year corrected to bring it to the middle of the year. It is now superseded by No. 2,¶ which was formed from the deaths occurring in the seven years, 1838–1844, and the Census of 1841. No. 3‖ was formed from the Census returns of 1841 and 1851 and the deaths occurring during the seventeen years, 1838–1854.

The Table known as the Government Males, constructed by Mr. John Finlaison** from the records of the National Debt Office, is sometimes used in combination with other Tables, but, I believe, never alone.

For much useful information respecting these Tables and their respective merits and defects, I must refer the reader to the "Encyclopædia Britannica," articles "Mortality" and "Annuities," both from the accurate pen of Mr. Milne; the Appendices to the Registrar-General's 5th, 8th, and 12th Reports; and Mr. A. G. Finlaison's Report on Government Annuitants.††

The different Rates of Interest used in the calculations are 3, 3½, and 4 per cent. Many reasons are given for their adoption; some persons considering that it should be the same as that upon which the premiums are calculated; and others contending that it should be the same as that realized by the Office or likely to be

* "Law of Mortality, deduced from the Combined Experience of 17 Offices, embracing 83,905 Policies, of which 40,616 are distinguished by denoting the Sex of the Lives assured, and by classing them into Town, Country, and Irish Assurances"; printed privately for the Offices in 1843.

See also, "Annuities and Assurances calculated from a New Rate of Mortality amongst Assured Lives, &c.," by Jenkin Jones. 1843.

† "Life Tables, founded upon the Discovery of a Numerical Law regulating the existence of every Human Being, &c.," by T. R. Edmonds, B.A. 1832.

‡ Philosophical Transactions of the Royal Society for the Year 1825.

§ Appendices to Registrar-General's 5th and 6th Reports.

¶ Appendix to Registrar-General's 12th Report.

‖ "English Life Tables, &c.," by W. Farr, M.D., F.R.S., D.C.L. 1864.

** "Report of John Finlaison, Actuary of the National Debt, on the Evidence and elementary Facts on which the Tables of Life Annuities are founded.—Ordered by the House of Commons to be printed, 31st March, 1829."

†† "Report and Observations on the Mortality of the Government Life Annuitants, by Alexander Glen Finlaison, Actuary of the National Debt. 1860.—Ordered by the House of Commons to be printed, 25th August, 1860."

realized in future. It is a doubtful question in many minds, whether the Offices will always be able to obtain 4 per cent. The rate of Interest has been falling for centuries, and there is little reason to doubt but that this depreciation of money will continue, though in a much slower degree. Again, as the funds of an Office accumulate, there will always be greater difficulty in finding investments for them at high rates of interest, combined with perfect security; and some of our oldest established Offices, with large accumulated funds, even now are not able to realize so large a rate as 4 per cent. The funds of an Assurance Company should be invested on none but the most perfect security, and cannot therefore be expected to yield much more than the rate at which the Government can borrow, increased slightly from the fact that those securities most frequently offered for its investments are not easily negociable in the open Market; and that it affords to borrowers many conveniences, for which they are not disinclined to pay a slightly increased rate, in the certainty that they will not be called upon to make frequent changes, which are sometimes very costly. It must nevertheless be remembered that the competition for these investments is increasing every year as Capital accumulates, which will tend to lower the rate of interest realizable upon them.

Before proceeding to examine the different methods pursued in the valuation of the liabilities of an Office under its policies, I will briefly state what appears to me to be the true principle upon which the valuation should be based. On the establishment of an Office it is assumed, or ought to be, that a certain Table will exactly correspond with the mortality likely to prevail among the persons whom they will insure; and that a certain rate of interest (generally 3 per cent) is what they may with certainty depend upon realizing. Under these circumstances, premiums calculated upon such data, increased by a small addition sufficient to cover all expenses likely to be incurred, are charged. We will leave out of consideration the addition made for bonus, as it is only charged to be returned again. Suppose now that all these conditions are realized, and that after a few years it is found that the mortality experienced corresponds exactly with the Table adopted; that the rate of interest realized is exactly 3 per cent; and that the additional charge, technically called the loading, has exactly covered all the expenses. How much ought the Office to have in hand? The answer naturally will be,—the total *net* premiums received in each year, less the claims paid in that year, accumulated at 3 per cent compound interest. This has been mathematically proved by Mr. Sprague in

the *Assurance Magazine*, vol. xi., p. 104, and practically illustrated by Mr. Meikle in the same volume, p. 245. Thus, if we suppose, for the sake of simplicity, that all who assured did so for £1, at the age x; and let n represent the number of years elapsed from the date of their entry to that of valuation, it can be easily proved, by the aid of those papers, that the amount in hand will be represented by

$$X\left\{\frac{N_{x-1}-N_{x+n-1}}{D_{x+n}}\cdot\frac{M_x}{N_{x-1}}-\frac{M_x-M_{x+n}}{D_{x+n}}\right\},^*$$

X being the number of policies then existing.

The amount standing to the credit of each person will be

$$\frac{N_{x-1}-N_{x+n-1}}{D_{x+n}}\cdot\frac{M_x}{N_{x-1}}-\frac{M_x-M_{x+n}}{D_{x+n}},$$

which Mr. Sprague has shown can be reduced to

$$1-\frac{1+a_{x+n}}{1+a_x},$$

the well known formula for the value of a policy.

This then *must* be the *true value*; for, all the conditions assumed at the outset remaining the same, it is impossible for the Office to have either more or less than this amount to the credit of each policy. If, however, the claims have not been so great as the Table originally employed would have led us to expect, or the rate of interest realized have been more than 3 per cent, the amount of funds in hand will be larger than shown above, and the difference might be fairly returned as realized profit; but whether there may be in consequence, any justification for altering the system of valuation, is a matter which will always require the gravest consideration. The Table used ought certainly to represent, if possible, the mortality that has been and will in future be experienced by the Office; and there might be many reasons for changing it, but this cannot be reasonably done until the Office has been in existence some years, because we know, for Mr. Farren† has clearly proved it, that selection has an effect on the rate of mortality during the first few years; and moreover the number of lives assured probably would not be sufficient to form a fair average. It used at one time to be considered that a Table showing a low rate of mortality would give a small valuation, but that has been found to be incorrect, the opposite being frequently the case, so that if the mortality expe-

* *Assurance Magazine*, vol. xi., p. 107, line 3; and p. 246, line 4.

† "The Chances of Premature Death and the Value of selection among Assured Lives." By E. J. Farren. 1850.

rienced has been very light, it can be no argument against an alteration.

Before the rate of interest is altered there should be a firm conviction that there is no likelihood of it ever falling below the rate proposed; for it is at the rate adopted that the reserve made must accumulate in order to meet all the claims. It will hereafter be seen that the increased return made to the assured in the shape of bonus, by the use of a high rate of interest, does not sufficiently compensate for the reduced amount in reserve; and that in after years, if the rate realized should continue to be the same, a valuation at the smaller rate would distribute as large, if not larger bonus, than a valuation made at the higher rate; while, on the other hand, if the rate realized ever fell below that adopted, it would be a very difficult matter to return to the lower one, and the assured in consequence might have to go without a bonus for some time. When it is proposed to alter the rate of interest, it should be considered in connection with the Table of Mortality used, since a 3 per cent valuation by one Table, as will be subsequently seen, may actually give a less reserve than a valuation made at 4 per cent by another Table.

There is another consideration also, which ought not to be passed over, and that is, that to alter the data at the valuation, is equal to declaring that the data upon which the premiums were originally calculated were wrong, and therefore, in justice to all the members present and future, the premiums ought to be re-calculated,—unless the Profits are returned in such a manner as to fully compensate for the difference.

In the case where an Assurance has been paid by a Single Premium we shall find, by a similar mode of reasoning as employed above, that, all the conditions remaining the same as at the outset, the Office will have in hand to the Credit of that Policy a sum equal to

$$\pi_{x+n}(1+a_{x+n})+\phi_x(1+a_{x+n}),$$

supposing the single premium charged to be $(\pi_x+\phi_x)(1+a_x)$, (where ϕ_x represents the addition made to the annual premium at age x for expenses); which value is made up of the sum necessary to meet the claims, and the amount necessary for future expenses, which cannot be distributed as realized profit but must be retained for the purposes for which it was originally charged. Therefore, although the value of the Policy may be truly represented by $\pi_{x+n}(1+a_{x+n})$, yet the amount the Office should have in reserve should be $(\pi_{x+n}+\phi_x)(1+a_{x+n})$, unless, of course, some of the usual annual expenses have been compounded, such as commission, &c.

We will now enter upon the consideration of the different methods at present employed, which I think are all included in the following:—

1st. That of valuing the sums assured and the pure Premiums only, by what is termed a true Table of Mortality, and at a true rate of interest.

2nd. The method which values the Premiums actually payable by means of a hypothetical Table of Annuities derived by an inverse process from the Office Premiums.

3rd. That of valuing by the Northampton Table and making a reserve, over and above that which such a valuation will give, of a proportion of the resulting surplus.

4th. Deducting the present value of the gross Premiums from the present value of the sums assured.

5th. That of deducting the present value of the gross premiums, less a certain percentage, from the present value of the sums assured.

6th, and lastly. A method, peculiar to those Offices which make annual valuations and return the surplus in the form of a percentage abatement of the next premium payable, which values the sums assured and the Office premiums (reduced by the proposed abatement) by a true Table of Mortality and rate of interest.

First Method:—*That of valuing the sums assured and the pure premiums only, by what is termed a true Table of Mortality, and a true rate of interest.*

This method, which has been variously termed "the true," "the pure," and "the net," method, is the one now most commonly used, and to which the remarks made above will fully apply. From what I have there stated it will be seen, that what is meant by a true Table is one exhibiting a rate of Mortality corresponding as nearly as possible to that prevailing amongst the lives assured; but there is so much diversity of opinion as to what is meant by a *true* rate of interest, that it is impossible to give a satisfactory definition of it. My own opinion is that it should be the rate assumed in the calculation of the premiums, for reasons previously stated. There is one Office, and there may perhaps be others, in which two different rates of Interest are used in the valuations, namely 4 per cent for some of the old policies and 3 per cent for the younger ones; the reason being that the premiums were originally calculated at 4 per cent, but afterwards altered and calculated at 3 per cent. Each policy is thus valued at the same rate as that at which the premium was calculated.

By this method the value of a Policy will be represented by the formula

$$(\pi_{x+n}-\pi_x)(1+a_{x+n}),$$

where x represents the age at which the Policy was taken out; n the number of years it has been in force at the valuation; a_x the value of an Annuity at age x; and π_x the pure annual premium at the same age, by the Table and rate of interest adopted.

There are three different ways of exhibiting the value of the liabilities by this method, all of which are used; thus, (1st), simply stating the value without its component parts, as

Dr.	*Cr.*
$(\pi_{x+n}-\pi_x)(1+a_{x+n})$	——

or, (2nd), exhibiting the values of the sums assured and the pure premiums, as

Dr.	*Cr.*
$\pi_{x+n}(1+a_{x+n})$	$\pi_x(1+a_{x+n})$

or, (3rd), showing the present value of the sums assured, the Office premiums and the loading, thus

Dr.	*Cr.*
$\pi_{x+n}(1+a_{x+n})+$ $\phi_x(1+a_{x+n})$	$(\pi_x+\phi_x)(1+a_{x+n})$.

At pages 265-282 I have given Tables of the values of Policies for £100, taken out at various ages and in force for various terms; by the different Tables of Mortality generally used and at the different rates of interest, whenever practicable. By a simple inspection of these it will be seen what variations there are even in the value of a single Policy. Let us take for example the following case: That of a Policy taken out at the age of 30 and valued at the end of 10, 20, 30, 40, 50, and 60 years. The values by the different Tables, according to the rate of interest used, are as follows:—

(*Age at entry*, 30.)

3 per Cent.							
Years in force.	Combined Experience.	Edmonds' Mean Mortality.	English Life, No. 2.	English Life, No. 3.	Carlisle.	Davies' Equitable.	Northampton.
10	12·677	12·439	12·506	12·691	11·746	11·691	11·573
20	28·592	27·244	27·568	27·930	25·562	26·247	25·030
30	46·092	45·344	45·246	44·951	44·100	41·369	39·864
40	62·971	62·347	61·951	61·977	60·484	57·658	56·845
50	76·877	75·881	75·452	75·422	73·903	74·029	73·320
60	87·877	85·246	84·406	84·117	82·978	85·232	84·406

3½ per Cent.				
Years in force.	Combined Experience.	English Life, No. 3.	Carlisle.	Davies' Equitable.
10	11·696	11·755	10·815	10·783
20	26·914	26·314	23·885	24·657
30	44·144	43·046	42·142	39·418
40	61·184	60·220	58·627	55·733
50	75·516	74·052	72·428	72·540
60	87·056	83·123	81·921	84·247

4 per Cent.							
Years in force.	Combined Experience.	English Life, No. 2.	English Life, No. 3.	Govern-ment Males.	Carlisle.	Davies' Equitable.	North-ampton.
10	10·793	10·700	10·896	9·770	9·959	9·953	10·049
20	25·333	24·389	24·798	24·430	22·311	23·175	22·293
30	42·268	41·491	41·219	40·955	40·270	37·561	36·389
40	59·440	58·448	58·504	57·430	56·815	53·862	53·357
50	74·163	72·700	72·693	71·748	70·967	71·067	70·578
60	86·225	82·435	82·128	81·726	80·864	83·258	82·525

Here we see that at 3 per cent the value of the Policy after 10 years will vary from £12·691 (the value by the English Life Table No. 3), to £11·573 (the value by the Northampton Table), being a difference of £1·118; and after 30 or 40 years it will vary to the extent of £6 and upwards, or from 10 to 15 per cent of the whole value. The difference between the values by the various tables is slightly less when the rate of interest is 3½ or 4 per cent; but the difference between the 3 per cent and 4 per cent values is very large, varying at the end of 10 years to the extent of £2·921; and at the end of 30 years the difference between the Northampton 4 per cent and Experience 3 per cent values is as much as £9·703, or upwards of 26½ per cent of the former.

Second Method:—*That which values the premiums actually payable by means of a hypothetical Table of annuities derived by an inverse process from the Office premiums.*

This method has been called "the fictitious," "the hypothetical," and "the reinsurance" method. It is now very seldom used, but was at one time universally adopted, probably under the impression that it invariably gave a larger value than when made by the pure method. That idea has been shown by Mr. Sprague, in a very neat mathematical demonstration, to be incorrect; and since this method has

received such a thorough examination by that gentleman, I cannot do better than refer the reader to his paper.* He there shows under what circumstances the value by this method is greater, equal, or less than by the net method; namely, if the loading be a constant addition, then the value is less; and if a constant percentage, the value is greater; while in those cases in which the loading is made up of a percentage and a further constant addition, if

$$\phi_x = k\pi_x + kd$$

(where d represents the discount upon £1 for one year and k is a constant quantity throughout), then the value by this method will be the same as by the net method; whence it is easy to perceive that in all cases where $\phi_x = k\pi_x + c$ that the value by the hypothetical method will be greater or less than by the net method, as

$$c < \text{ or } > kd.$$

His remarks on pages 102, 103 (vol. xi.), with reference to the unscientific nature of this method are so clear and to the point, that I feel bound to quote them here. He says, "It appears to me to "have a rigidity about it which unfits it for general use. Thus, "if some of the policies are taken at lower premiums than those "charged by the Company usually, which is sometimes the case in "reassurances, all such policies will cause a considerable loss on "the first valuation made. On the other hand, if the business of "another Company be taken over, in which higher premiums are "charged, all the recent policies will appear in the valuation as "assets instead of liabilities; and the values of all will be smaller "than those given by the more usual methods.

"So also there will be a difficulty if the Company wishes to "reduce or increase its rates: if the rates are reduced, and the old "policies are valued by the annuities deduced from the new pre-"miums, a large fictitious profit will accrue, to avoid which we "must have recourse to the very unscientific process of valuing the "two sets of policies, issued before and after the change of rates, "by different Annuity Tables. The opposite effect would follow if "the premiums were considered insufficient, and were raised," unless the new premiums bore the same relation to the old ones as

$$\pi''_x + d = (1+k)(\pi'_x + d),$$

where π'_x denotes the old premium, and π''_x the new one, at the age x.

* "On certain Methods proposed for the Valuation of the Liabilities of a Life Assurance Company," by Thomas Bond Sprague, M.A.; *Assurance Magazine*, vol. xi., p. 90.

Mr. Tucker, who gave to this method the name of "reinsurance," and who so strongly upheld and advocated it in a paper read before the Institute of Actuaries in November, 1862,* argues that the reserve made for a policy taken out at age x and valued n years after, should be sufficient to provide for the difference between the premium now paid and "such a sum as another Office charging "similar premiums would be willing to undertake the liabilities "for," that is the Office premium at age $(x+n)$.

This may appear reasonable if the percentage for commission, which the Office does not get and which the reassuring Office will allow, be previously deducted; and if the bonus be divided in proportion to the loading (less commission) paid since the last division accumulated at the interest realized; always allowing of course for the increase in the loading (if any) allowed for at the last valuation. Such a plan would, in most cases, have the advantage now frequently sought of equitably giving an increasing bonus at successive divisions.

The value of a policy under these circumstances—the commission being 5 per cent—will be

$$\left(\frac{19}{20}\pi'_{x+n}-\frac{19}{20}\pi'_x\right)(1+a_{x+n})=\frac{19}{20}(\pi'_{x+n}-\pi'_x)(1+a_{x+n}).$$

In the case of an Office—like the Eagle for instance—loading the premiums with a constant addition and one-nineteenth, so that $\pi'_x=\frac{20}{19}(\pi_x+c)$, this value will become

$$(\pi_{x+n}-\pi_x)(1+a_{x+n}),$$

—the same as the value by the net method.

Mr. Tucker says that the value should be

$$(\pi'_{x+n}-\pi'_x)(1+a'_{x+n}),$$

"For then the contribution at age $(x+n)$ is,

"The original premium $\quad \pi'_x,$

"and the reserve $\quad \pi'_{x+n}-\pi'_x,$

"that is $\quad \pi'_{x+n}=\pi_{x+n}+k\pi_{x+n}.$"

But unless he assumes that the Mortality the Office will experience in future, will be the same as that deduced from the Office

* "On the Proper Mode of estimating the Liabilities of Life Insurance Companies," by Robert Tucker, &c. &c.; *Assurance Magazine*, vol. x., p. 312.

premiums—which surely cannot be his meaning—the contribution from the reserve will be, not $\pi'_{x+n}-\pi'_{x}$, but

$$(\pi'_{x+n}-\pi'_{x})\frac{1+a'_{x+n}}{1+a_{x+n}},$$

a smaller quantity, and therefore the whole contributions will be less than π'_{x+n}.

The title of "reinsurance" cannot therefore be justly claimed for this method.

I have considered this method to be a question of such great importance that I have constructed four Tables of the values of Policies by it, which will be found at pages 283-286. The first exhibits the values when the premiums are formed from the Carlisle 3 per cent, with a loading of a constant addition of ·0037 to the premium for £1, being the average difference between those premiums and the premiums generally charged on the "non-profit" scale. The second exhibits the values when the Carlisle 3 per cent premiums are loaded with a constant percentage of 13 per cent; and the third represents the values when the premiums are loaded 25 per cent, being the average difference per cent between the Carlisle premiums and those generally charged on the "non-profit" and "with-profit" scales, respectively. The fourth Table shows the values formed from the average of the premiums of 6 Offices, namely, the Scottish Provident, Pelican, Crown, Eagle, Guardian, and Law Life Offices.

Third Method:—*That of valuing by the Northampton Table and making a reserve, over and above that which such a valuation will give, of a proportion of the resulting surplus.*

This method has been classed with the previous one (No. 2), by Mr. Jellicoe,* who has bestowed great attention upon it; but, besides the introduction of another element into the reserve, there are some Offices using this method which do not charge the Northampton pure premiums, and consequently cannot be said to value by tables formed from the Office premiums. I have therefore considered it as distinct from the previous method.

The Offices who used to value by this method were very numerous; but they are fast abandoning it, and the number who still cling to it is very small. It is impossible to say what the reserve by this method will be, since the surplus will depend so much upon the management of the Office. We may however get some idea of

* "On the Objectionable Character of certain methods very generally adopted for the Determination and Division of Surplus in Life Assurance Companies." *Assurance Magazine*, vol. iii., page 165.

what it is in practice from the lately published experience of an old established Office, which has been in the habit of making large returns to the assured as bonus, and which used to value by this method, using 4 per cent as the rate of interest, and reserving one-third of the surplus, but changed its method on the last occasion, adopting instead the Carlisle 3 per cent net method. The results produced by the two methods are stated to be almost identical.

In the employment of this method both 3 per cent and 4 per cent are the rates of interest used; and from what has been stated above I think we may safely say, that the values by this method will range from those shown by the Northampton Table at 4 per cent to values slightly greater than exhibited by the Carlisle Table at 3 per cent.

There is one Office which slightly modifies the working of this plan by previously deducting a small percentage from the premiums, and thus making a larger reserve. Although the reserve made may be generally sufficient, still an Office using this method is placed in the false position of exhibiting increased security where none exists, by showing a large reserve (which is wholly fictitious) out of what is represented to be actual surplus; while on the other hand, sueh a system often gives rise to the belief in the minds of the Assured of large accumulations of undivided profits, created by their payments, but in which they may never participate.

Fourth Method:—*That of deducting the present value of the gross premiums from the present value of the Sums Assured.*

I cannot say for certain that this method is ever now adopted except in the modified form explained in the 6th Method. By adopting such a method an Office would set down as an asset, sums which it never possessed and which it might never receive, while it would leave nothing to meet the expenses in future years, which consequently would have to be paid out of the surplus, if any, of interest realized over that assumed. Under this system the policies for some years would be represented as assets instead of liabilities. It is a method possessing no principle whatever.

At page 287 I have given the values of Policies by this method, assuming the premiums charged to be the Carlisle 3 per cent with a loading of 25 per cent, and using the Carlisle Table and 3 per cent interest as the basis of the calculations.

Fifth Method:—*That of deducting the present value of the gross premiums, less a certain percentage, from the present value of the sums assured.*

The principle upon which this method is founded is, I believe,

that certain premiums having been charged by the Office, and a certain proportion being necessary to meet the expenses and perhaps to provide for future bonuses, the remainder of the premiums may be considered as the portion which the Office charges to meet all the liabilities under its policies. The values by this method may in consequence be greater or less than those given by the net method, according as the amount deducted is greater or less than the loading. But besides this there is a great want of consistency about the system; for, if the premiums are loaded with a constant addition, or a constant and a percentage, which is nearly always the case, then the values of those entering at the older ages will be greater in proportion than those entering at the younger ages; and it might happen that the reserve made for the older lives would be more than sufficient, while that for the younger lives would not be enough, as tested by comparison with the reserve required by the true method.

Almost every Office has a class of policy-holders who do not share in the profits, and who, in consequence, pay a smaller premium than the others. Their policies must appear in the valuation as of much greater value than those entitled to participate in the profits, which is quite at variance with all ideas of equity; for, if there ought to be any difference in the reserve made for the two classes, it certainly ought to be in favour of that which expects to receive large returns in the future.

At pages 289 and 290 I have given the values of the sums assured and the premiums payable (supposing them to be formed from the Carlisle 3 per cent with a loading of 25 per cent), from which it will be easy to find what result any particular reduction will have.

Sixth Method:—*A method peculiar to those Offices which make annual valuations and return the surplus in the form of a percentage abatement of the next premium payable, which values the sums assured and the Office premiums (reduced by the proposed abatement) by a true Table of Mortality and rate of interest.*

The Offices which adopt this method are rather popular in the present day, and, in most cases, deservedly so; for they are generally conducted upon principles of such strict economy and otherwise managed with such care and ability that they may be classed amongst the soundest of these Institutions. It will not, therefore, I think, be out of place if we give some attention to the exact mode by which they value their liabilities; but first I will observe that these Offices are invariably mutual; that in some of them—those giving the largest reductions more especially—no agency fees or

commission are allowed, and also that no reduction is made until after a certain number of premiums (generally five or seven) have been paid.*

In the first place, the valuation is made according to the fourth method above explained, that is, by valuing the sums assured and gross premiums, thus giving a certain surplus. Then from the value of the gross premiums is deducted the present value of all the full premiums to be received from those members not yet entitled to any abatement, which may be represented by the formula

$$S.\pi'_x(1+a_{x+n\overline{6-n|}}),$$

(supposing no reduction allowed before seven premiums are paid, and S to represent their total number), then the ratio this difference bears to the surplus as above obtained will be the proportion to be deducted from the next annual premiums payable. The reserve made is therefore, supposing the Office to have no expenses besides what can be provided for out of surplus interest and miscellaneous profit, such a sum as accumulated at the interest assumed in the calculation, will be sufficient, with the due payment in future of the premiums *reduced as above explained,* to meet all the claims as they arise.

It is above supposed that the Offices insure none but members entitled to an abatement; but in some cases they insure others at a smaller premium who are not so entitled. When this is the case, the value of their policies is the same as that by the 4th Method; and such policies will in consequence appear for the first few years as an asset, unless they are valued by a different process.

If it were impossible to increase the premiums after they had been once reduced, this method would have to be classed with the 4th Method; but it is not so, for a Member is always liable to pay to the extent of the full premium, and if in any year the business has been less profitable than usual, the reduction may not amount to the average of former years.

This method might well be called the *elastic* method, for the reserve will expand or contract according as the business of the year proves good or bad. If the reduction be smaller than in previous years there will be a corresponding reduction in the reserve; while, if it be larger, there will be a corresponding increase in the reserve;—and the only limits above or below which

* We believe that the statements contained in this paragraph require some correction, as the method of valuation in question is more generally used than Mr. Manly supposes; and is applicable, with proper modifications, to proprietary as well as mutual Offices.—Ed. *J. I. A.*

it cannot go, are, on the one hand, the present value of the sums assured, and on the other hand, the present value of the sums assured less that of the gross premiums.

At page 288 will be found a Table of the values of policies according to the Carlisle rate of mortality and 3 per cent interest, the premiums chargeable being those formed from the Carlisle 3 per cent with 25 per cent loading and reduced after 5 payments by 50 per cent. In practice the rate of interest generally used is 4 per cent; but since I have made the Carlisle 3 per cent values the standard for comparison in this Essay I considered it advisable to use that rate in this case.

The following Tables will exhibit at a glance the difference in the value of a single policy of £100 according to the various Tables of mortality and rates of interest now generally used, and by the different methods above explained. I have not thought it necessary to multiply the latter Tables, since a comparison of the different methods by the use of one Table of mortality only will be quite sufficient for the purpose of showing their several effects.

Table No. 1 (Carlisle 3 per cent values) was taken from Mr. B. Hall Todd's "Life Assurance Investigation Tables." Tables 4 and 5 were originally taken from W. E. Hillman's "Tables of the Value of a Policy according to the Mortality indicated by the Carlisle Observations and the Combined Experience," but I afterwards calculated the values myself, and altered them whenever they were found to differ by more than unity in the last figure. Table No. 6 was also taken from Mr. Hillman's work, and corrected by Mr. Laundy's "List of errors" in the *Assurance Magazine*, vol. ix., page 240. I can fully endorse Mr. Laundy's remark, that no reliance whatever can be placed upon the correctness of the values in Hillman's tables.

Table No. 17 was taken from "Jones on Reversionary Payments," page 937, vol. 2, which appears to have been copied from Mr. Griffith Davies' "Table No. 44." All the other Tables I have myself carefully computed and checked; a few errors may possibly have crept in, but I think none that would seriously affect any results hereafter produced by their use.

TABLE No. 1.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to the Carlisle Table of Mortality, reckoning Interest at 3 per Cent.*

Age when Assured.	Number of years elapsed.								
	5	10	15	20	25	30	35	40	45
15	3·760	8·124	12·828	17·591	23·066	28·493	35·110	43·140	51·271
20	4·534	9·422	14·371	20·061	25·699	32·575	40·919	49·366	56·302
25	5·120	10·304	16·264	22·169	29·372	38·113	46·961	54·226	62·507
30	5·464	11·746	17·970	25·562	34·773	44·100	51·757	60·484	68·322
35	6·644	13·228	21·259	31·003	40·869	48·968	58·200	66·491	72·395
40	7·053	15·655	26·092	36·660	45·336	55·226	64·106	70·430	76·689
45	9·255	20·484	31·854	41·188	51·828	61·383	68·187	74·920	79·252
50	12·374	24·904	35·190	46·914	57·445	64·941	72·362	77·137	75·448
55	14·299	26·038	39·419	51·435	59·990	68·459	73·906	71·981	79·995
60	13·698	29·310	43·332	53·315	63·196	69·554	67·305	76·655	
65	18·091	34·338	45·905	57·355	64·722	62·117	72·951		
70	19·835	33·958	47·936	56·930	53·750	66·976			
75	17·617	35·054	46·272	42·307	58·806				
80	21·174	34·781	29·972						
85	17·262	11·161							
90	(—)7·374								
Age when Assured.	**Number of years elapsed.**								
	50	**55**	**60**	**65**	**70**	**75**	**80**		
15	57·946	65·554	72·386	77·250	82·067	85·162	84·068		
20	64·207	71·305	76·359	81·365	84·582	83·445			
25	69·944	75·237	80·480	83·850	82·659				
30	73·903	79·427	82·978	81·723					
35	78·237	81·995	80·667						
40	80·714	79·292							
45	77·721								
50	82·469								

TABLE No. 2.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to the Carlisle Table of Mortality, reckoning interest at 3½ per Cent.*

Age when [illegible]	5	10	1
15	3·337	7·293	11·
20	4·093	[illegible]	13·
25	4·661	9·406	14·
30	4·979	10·815	16·
35	6·142	12·264	19·
40	6·522	[illegible]	24·
45	8·700	19·497	30·
50	11·824	23·987	34·
55	13·793	25·169	38·
60	13·196	[illegible]	42·
65	17·621	33·642	45·
70	19·448	[illegible]	47·
75	17·268	[illegible]	45·
80	20·880	34·481	29·
85	17·127	[illegible]	
90	(−)7·472		

Age when Assured.	50	55	6
15	55·611	63·433	70·
20	62·170	69·528	74·
25	68·227	73·714	79·
30	72·428	78·185	81·
35	77·042	80·974	79·
40	79·729	78·214	
45	76·695		

TABLE No. 3.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to the Carlisle Table of Mortality, reckoning Interest at 4 per Cent.*

Age when Assured.	Number of years elapsed.								
	5	10	15	20	25	30	35	40	45
15	2·970	6·565	10·539	14·604	19·449	24·304	30·499	38·361	46·564
20	3·704	7·798	11·988	16·982	21·986	28·371	36·474	44·928	51·931
25	4·254	8·603	13·789	18·986	25·616	34·031	42·810	50·083	58·651
30	4·542	9·959	15·388	22·311	31·100	40·270	47·865	56·815	65·051
35	5·674	11·360	18·614	27·821	37·428	45·384	54·759	63·388	69·585
40	6·028	13·718	23·479	33·662	42·098	52·038	61·186	67·756	74·399
45	8·184	18·571	29·407	38·384	48·961	58·696	65·688	72·757	77·384
50	11·313	23·117	32·893	44·412	55·015	62·629	70·328	75·368	73·507
55	13·310	24·332	37·321	49·276	57·862	66·543	72·226	70·128	
60	12·715	27·700	41·489	51·394	61·408	67·962	65·542		
65	17·166	32·965	44·312	55·785	63·295	60·522			
70	19·073	32·772	46·622	55·688	52·340				
75	16·927	34·042	45·244	41·108					
80	20·602	34·087	29·107						
85	16·984	10·712							
90	(−)7·555								
Age when Assured.	Number of years elapsed.								
	50	55	60	65	70	75	80		
15	53·359	61·366	68·735	74·027	79·378	82·880	81·587		
20	60·183	67·777	73·232	78·747	82·356	81·023			
25	66·538	72·202	77·929	81·678	80·294				
30	70·967	76·949	80·864	79·418					
35	75·851	79·953	78·438						
40	78·747	77·141							
45	75·675								

Table No. 4.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to the Experience Table of Mortality, reckoning Interest at 3 per Cent.*

Age when Assured.	Number of years elapsed.								
	5	10	15	20	25	30	35	40	45
15	3·537	7·579	12·183	17·400	23·315	30·012	37·291	44·902	52·659
20	4·189	8·962	14·370	20·503	27·447	34·991	42·882	50·924	58·845
25	4·981	10·627	17·027	24·274	32·149	40·385	48·777	57·046	64·816
30	5·941	12·677	20·304	28·592	37·260	46·092	54·795	62·971	70·377
35	7.161	15·271	24·082	33·297	42·687	51·939	60·632	68·506	75·416
40	8·735	18·225	28·152	38·266	48·232	57·595	66·076	73·520	80·042
45	10·399	21·276	32·358	43·277	53·537	62·830	70·984	78·132	84·788
50	12·319	24·507	36·694	48·144	58·516	67·618	75·594	83·023	89·258
55	14·077	27·947	40·980	52·784	63·144	72·222	80·677	87·774	
60	16·142	31·310	45·048	57·106	67·671	77·512	85·770		
65	18·088	34·470	48·849	61·447	73·183	83·031			
70	20·000	37·554	52·934	67·261	79·284				
75	21·942	41·168	59·076	74·105					
80	24·630	47·572	66·826						

Age when Assured.	Number of years elapsed.								
	50	55	60	65	70	75	80		
15	60·301	67·482	73·986	79·694	84·695	89·354	93·264		
20	66·289	73·032	78·949	84·134	88·963	93·017			
25	71·852	78·029	83·440	88·481	93·711				
30	76·877	82·572	87·877	92·329					
35	81·471	87·111	91·845						
40	86·117	91·216							
45	90·375								

TABLE No. 5.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to the Experience Table of Mortality, reckoning Interest at 3½ per Cent.*

Age when Assured.	Number of years elapsed.								
	5	10	15	20	25	30	35	40	45
15	3·113	6·729	10·923	15·760	21·341	27·783	34·896	42·441	50·245
20	3·732	8·059	13·054	18·814	25·464	32·803	40·592	48·646	56·676
25	4·496	9·683	15.665	22·573	30·198	38·288	46·655	54·998	62·929
30	5·431	11·696	18·930	26·914	35·385	44·144	52·879	61·184	68·785
35	6·625	14·274	22·716	31·673	40·937	50·173	58·955	66·992	74·109
40	8·192	17·234	26·825	36·747	46·638	56·042	64·650	72·272	79·010
45	9·849	20·296	31·103	41·876	52·121	61·495	69·799	77·137	84·034
50	11·590	23·576	35·528	46·890	57·290	66·499	74·639	82·290	88·750
55	13·559	27·076	39·928	51·690	62·108	71·314	79·968	87·276	
60	15·639	30·507	44·114	56·165	66·816	76·826	85·280		
65	17·624	33·755	48·039	60·664	72·530	82·551			
70	19·581	36·921	52·248	66·653	78·818				
75	21·562	40·620	58·534	73·661					
80	24·297	47·135	66·420						

Age when Assured.	Number of years elapsed.								
	50	55	60	65	70	75	80		
15	58·026	65·423	72·194	78·189	83·489	88·470	92·676		
20	64·313	71·300	77·488	82·958	88·099	92·441			
25	70·188	76·616	82·297	87·638	92·148				
30	75·516	81·464	87·056	91·778					
35	80·400	86·313	91·306						
40	85·341	90·689							
45	89·858								

Table No. 6.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to the Experience Table of Mortality, reckoning Interest at 4 per Cent:*

Age when Assured.	Number of years elapsed.								
	5	10	15	20	25	30	35	40	45
15	2·735	5·976	9·791	14·270	19·527	25·708	32·644	40·104	47·921
20	3·330	7·255	11·861	17·263	23·618	30·748	38·420	46·455	54·579
25	4·058	8·822	14·413	20·987	28·363	36·297	44·611	53·013	61·087
30	4·966	10·793	17·645	25·333	33·603	42·268	51·026	59·440	67·212
35	6·131	13·340	21·429	30·133	39·249	48·467	57·320	65·498	72·813
40	7·681	16·299	25·570	35·283	45·100	54·533	63·245	71·037	77·978
45	9·335	19·379	29·898	40·532	50·751	60·188	68·628	76·146	83·274
50	11·078	22·680	34·411	45·679	56·087	65·398	73·690	81·552	88·241
55	13·048	26·240	38·913	50·618	61·087	70·412	79·254	86·776	
60	15·170	29·746	43·206	55·247	65·972	76·140	84·791		
65	17·182	33·050	47·244	59·887	71·873	82·071			
70	19·161	36·299	51·565	66·038	78·352				
75	21·200	40·085	57·988	73·221					
80	23·965	46·685	66·016						

Age when Assured.	Number of years elapsed.								
	50	55	60	65	70	75	80		
15	55·821	63·411	70·422	76·693	82·278	87·574	92·079		
20	62·382	69·590	76·037	81·780	87·224	91·857			
25	68·542	75·211	81·152	86·784	91·576				
30	74·163	80·355	86·225	91·220					
35	79·328	85·505	90·761						
40	84·558	90·157							
45	89·338								

TABLE No. 7.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to the English Life Table No. 3 (Males), reckoning Interest at 3 per Cent.*

Age when Assured.	Number of years elapsed.								
	5	10	15	20	25	30	35	40	45
15	4·518	8·716	13·378	18·588	24·371	30·707	37·571	44·788	52·315
20	4·397	9·278	14·735	20·792	27·428	34·617	42·175	50·059	58·000
25	5·106	10·814	17·149	24·091	31·610	39·515	47·762	56·069	63·919
30	6·015	12·691	20·006	27·930	36·261	44·951	53·705	61·977	69·299
35	7·103	14·887	23·318	32·181	41·428	50·742	59·544	67·335	73·849
40	8·379	17·455	26·996	36·950	46·976	56·451	64·837	71·849	77·459
45	9·906	20·320	31·184	42·126	52·468	61·621	69·275	75·398	80·145
50	11·559	23·617	35·763	47·242	57·401	65·897	72·693	77·962	81·965
55	13·634	27·368	40·347	51·834	61·439	69·124	75·082	79·608	83·013
60	15·901	30·929	44·230	55·352	64·250	71·148	76·389	80·331	
65	17·869	33·685	46·910	57·490	65·693	71·925	76·612		
70	19·257	35·359	48·241	58·228	65·816	71·523			
75	19·942	35·897	48·266	57·664	64·732				
80	19·929	35·379	47·118	55·947					
85	19·296	33·956	44·983						
90	18·166	31·828							
95	16·696								

Age when Assured.	Number of years elapsed.								
	50	55	60	65	70	75	80	85	
15	59·898	67·064	73·406	78·710	82·953	86·242	88·741	90·621	
20	65·505	72·148	77·702	82·146	85·591	88·208	90·177		
25	70·867	76·677	81·325	84·928	87·666	89·725			
30	75·422	80·320	84·117	87·002	89·173				
35	79·060	83·101	86·171	88·480					
40	81·809	85·113	87·599						
45	83·752	86·465							
50	84·976								

TABLE No. 8.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to the English Life Table No. 3 (Males), reckoning Interest at 3½ per Cent.*

Age when Assured.	Number of years elapsed.								
	5	10	15	20	25	30	35	40	45
15	4·098	7·890	12·163	17·016	22·488	28·579	35·277	42·417	49·973
20	3·954	8·410	13·470	19·176	25·527	32·511	39·957	47·835	55·877
25	4·639	9·908	15·849	22·461	29·733	37·485	45·688	54·061	62·066
30	5·525	11·755	18·689	26·314	34·444	43·046	51·826	60·220	67·724
35	6·595	13·934	22·005	30·610	39·715	49·009	57·894	65·837	72·534
40	7·857	16·499	25·711	35·459	45·409	54·921	63·425	70·595	76·371
45	9·378	19·377	29·955	40·753	51·077	60·306	68·087	74·356	79·244
50	11·033	22·706	34·622	46·014	56·198	64·785	71·702	77·096	81·214
55	13·121	26·515	39·319	50·766	60·418	68·193	74·256	78·884	82·377
60	15·416	30·155	43·331	54·440	63·389	70·368	75·695	79·716	
65	17·425	33·002	46·136	56·717	64·967	71·265	76·019		
70	18·864	34·770	47·583	57·575	65·201	70·959			
75	19·604	35·396	47·711	57·111	64·207				
80	19·643	34·961	46·653	55·479					
85	19·062	33·612	44·595						
90	17·977	31·547							
95	16·545								

Age when Assured.	Number of years elapsed.								
	50	55	60	65	70	75	80	85	
15	57·685	65·059	71·650	77·208	81·685	85·176	87·841	89·852	
20	63·566	70·439	76·234	80·902	84·543	87·321	89·419		
25	69·222	75·255	80·116	83·906	86·799	88·983			
30	74·052	79·149	83·123	86·157	88·447				
35	77·929	82·136	85·348	87·772					
40	80·875	84·313	86·908						
45	82·975	85·792							
50	84·322								

TABLE No. 9.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to the English Life Table No. 3 (Males), reckoning Interest at 4 per Cent.*

Age when Assured.	Number of years elapsed.								
	5	10	15	20	25	30	35	40	45
15	3·728	7·158	11·075	15·592	20·765	26·607	33·126	40·172	47·729
20	3·562	7·631	12·324	17·696	23·764	30·537	37·855	45·705	53·822
25	4·219	9.085	14·656	20·948	27·971	35·560	43·700	52·116	60·255
30	5·080	10·896	17·466	24·798	32·721	41·219	50·007	58·504	66·173
35	6·127	13·049	20·773	29·121	38·073	47·331	56·283	64·363	71·231
40	7·373	15·602	24·494	34·031	43·893	53·430	62·037	69·354	75·289
45	8·883	18·483	28·780	39·427	49·723	59·015	66·914	73·322	78·346
50	10·536	21·837	33·522	44·821	55·019	63·688	70·721	76·235	80·463
55	12·631	25·693	38·323	49·722	59·412	67·272	73·436	78·162	81·741
60	14·950	29·406	42·453	53·544	62·541	69·595	75·005	79·102	
65	16·997	32·337	45·378	55·956	64·251	70·611	75·428		
70	18·481	34·193	46·937	56·930	64·593	70·397			
75	19·274	34·907	47·166	56·566	63·685				
80	19·366	34·552	46·196	55·015					
85	18·833	33·273	44·210						
90	17·791	31·266							
95	16·391								

Age when Assured.	Number of years elapsed.								
	50	55	60	65	70	75	80	85	
15	55·544	63·100	69·920	75·717	80·420	84·107	86·935	89·076	
20	61·671	68·755	74·777	79·662	83·492	86·429	88·653		
25	67·601	73·845	78·910	82·882	85·927	88·234			
30	72·693	77·981	82·128	85·308	87·716				
35	76·803	81·171	84·521	87·058					
40	79·942	83·511	86·214						
45	82·198	85·116							
50	83·665								

TABLE No. 10.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to the English Life Table No. 2 (Males), reckoning Interest at 3 per Cent.*

Age when Assured.	Number of years elapsed.								
	5	10	15	20	25	30	35	40	45
15	4·259	8·406	13·040	18·204	23·915	30·177	37·013	44·499	52·386
20	4·332	9·171	14·565	20·530	27·071	34·211	42·031	50·268	58·088
25	5·058	10·697	16·932	23·769	31·232	39·406	48·016	56·190	63·876
30	5·939	12·506	19·707	27·568	36·177	45·246	53·856	61·951	69·239
35	6·982	14·638	22·995	32·148	41·789	50·942	59·549	67·297	73·902
40	8·231	17·215	27·055	37·420	47·260	56·513	64·843	71·943	77·967
45	9·790	20·512	31·807	42·530	52·612	61·689	69·426	75·696	80·578
50	11·886	24·406	36·293	47·470	57·532	66·109	73·059	78·471	82·572
55	14·209	27·699	40·384	51·803	61·537	69·425	75·566	80·221	
60	15·725	30·509	43·821	55·166	64·361	71·519	76·945		
65	17·544	33·338	46·801	57·711	66·205	72·643			
70	19·155	35·482	48·713	59·015	66·823				
75	20·195	36·561	49·304	58·961					
80	20·507	36·475	48·576						

Age when Assured.	Number of years elapsed.								
	50	55	60	65	70	75	80		
15	59·873	66·913	73·251	78·653	83·031	86·439	89·022		
20	65·441	72·061	77·703	82·276	85·836	88·534			
25	70·795	76·693	81·473	85·195	88·015				
30	75·452	80·486	84·406	87·376					
35	79·254	83·421	86·579						
40	82·177	85·572							
45	84·278								

TABLE No. 11.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to the English Life Table No. 2 (Males), reckoning Interest at 4 per Cent.*

Age when Assured.	Number of years elapsed.								
	5	10	15	20	25	30	35	40	45
15	3·473	6·847	10·731	15·195	20·282	26·032	32·503	39·822	47·769
20	3·496	7·519	12·143	17·414	23·371	30·074	37·656	45·890	53·886
25	4·169	8·961	14·423	20·595	27·541	35·398	43·930	52·215	60·180
30	5·000	10·700	17·140	24·389	32·588	41·491	50·137	58·448	66·081
35	5·999	12·779	20·409	29·040	38·411	47·512	56·261	64·296	71·263
40	7·213	15·329	24·511	34·480	44·162	53·469	62·017	69·429	75·524
45	8·748	18·643	29·387	39·822	49·852	59·064	67·052	73·621	78·801
50	10·844	22·618	34·053	45·045	55·140	63·894	71·092	76·769	81·173
55	13·207	26·032	38·361	49·684	59·503	67·577	73·944	78·883	
60	14·777	28·982	42·028	53·340	62·643	69·979	75·670		
65	16·668	31·976	45·250	56·166	64·773	71·451			
70	18·370	34·299	47·398	57·727	65·741				
75	19·514	35·560	48·215	58·031					
80	19·937	35·659	47·856						

Age when Assured.	Number of years elapsed.								
	50	55	60	65	70	75	80		
15	55·487	62·907	69·721	75·629	80·488	84·320	87·292		
20	61·572	68·631	74·752	79·786	83·756	86·835			
25	67·495	73·838	79·054	83·167	86·358				
30	72·700	78·143	82·435	85·764					
35	76·992	81·510	85·015						
40	80·330	84·059							
45	82·820								

Table No. 12.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to Davies' Equitable Table, reckoning Interest at 3 per Cent.*

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Table No. 13.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to Davies' Equitable Table, reckoning Interest at 3½ per Cent.*

Age wh Assure
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90

Age wh Assure
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45

Table No. 14.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to Davies' Equitable Table, reckoning Interest at 4 per Cent.*

e when sured.	5	10
15	3·515	7·268
20	3·889	8·012
25	[illegible]	8·799
30	4·711	9·953
35	5·501	11·989
40	6·866	14·683
45	8·394	16·814
50	9·191	18·726
55	10·500	21·975
60	12·821	26·107
65	15·240	30·779
70	18·333	37·289
75	23·212	42·442
80	25·044	42·137
85	22·804	60·141
90	48·367	

e when sured.	50	55
15	51·688	59·051
20	57·559	65·340
25	63·937	72·308
30	71·067	78·313
35	77·240	82·431
40	81·408	90·400
45	89·693	

TABLE No. 15.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to Edmonds' Mean Mortality Table, reckoning Interest at 3 per Cent.*

Age when Assured.	Number of years elapsed.								
	5	10	15	20	25	30	35	40	45
15	4·262	8·796	13·628	18·799	24·372	30·442	37·159	44·761	52·792
20	4·735	9·783	15·184	21·005	27·345	34·362	42·302	50·691	58·671
25	5·298	10·968	17·078	23·733	31·099	39·434	48·240	56·617	64·342
30	5.987	12·439	19·467	27·244	36·046	45·344	54·190	62·347	69·620
35	6·863	14·338	22·611	31·973	41·863	51·272	59·949	67·685	74·345
40	8·026	16·908	26·960	37·579	47·682	56·998	65·304	72·454	78·393
45	9·657	20·587	32·132	43·117	53·245	62·276	70·050	76·507	81·679
50	12·098	24·877	37·036	48·247	58·243	66·849	73·996	79·721	84·144
55	14·539	28·370	41·124	52·497	62·286	70·417	76·930	81·961	
60	16·185	31·109	44·415	55·871	65·385	73·005	78·893		
65	17·806	33·682	47·349	58·701	67·793	74·817			
70	19·316	35·944	49·754	60·816	69·362				
75	20·609	37·725	51·435	62·027					
80	21·560	38·829	52·170						
85	22·015	39·023							
90	21·809								

Age when Assured.	Number of years elapsed.								
	50	55	60	65	70	75	80		
15	60·433	67·478	73·760	79·167	83·659	87·256	90·036		
20	66·030	72·592	78·240	82·931	86·689	89·592			
25	71·229	77·158	82·083	86·028	89·075				
30	75·881	81·081	85·246	88·464					
35	79·876	84·306	87·729						
40	83·150	86·825							
45	85·675								

TABLE No. 16.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to John Finlaison's Government Annuity Tables (Males), reckoning Interest at 4 per Cent.*

Age wh[illegible] Assure[illegible]	10[illegible]
15	5·0
20	4·3
25	7·0
30	9·7
35	12·2
40	16·2
45	19·3
50	21·6
55	25·3
60	27·9
65	29·4
70	33·6
75	37·4
80	35·3
85	

Age wh[illegible] Assure[illegible]	5[illegible]
15	60·[illegible]
20	65·[illegible]
25	72·[illegible]
30	77·[illegible]
35	80·[illegible]
40	·

TABLE No. 17.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to the Northampton Table, reckoning Interest at 3 per Cent.*

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85
90

...e when ...sured.		
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40	82·365	92·158
45	91·542	

Table No. 18.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to the Northampton Table, reckoning Interest at 4 per Cent.*

Age when Assured.	5	10	15
15	4·254	7·598	11·29
20	3·492	7·349	11·70
25	3·996	8·509	13·64
30	4·701	10·049	15·83
35	5·612	11·688	18·46
40	6·437	13·612	21·10
45	7·668	15·677	24·41
50	8·674	18·140	28·56
55	10·366	21·779	34·27
60	12·733	26·674	40·61
65	15·975	31·946	46·99
70	19·007	36·920	51·86
75	22·116	40·570	53·74
80	23·694	40·606	73·29
85	22·164	64·996	
90	55·029		

Age when Assured.	50	55	60
15	50·756	58·623	66·48
20	56·785	64·999	72·74
25	63·732	71·753	78·44
30	70·578	77·549	82·52
35	76·441	81·663	91·75
40	80·572	91·263	
45	90·662		

TABLE No. 19.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to a Hypothetical Table formed from Carlisle 3 per Cent Premiums, with loading of constant addition of ·0037 to the premium for £1.*

Age when Assured.	5	10
15	3·469	7·522
20	4·198	8·755
25	4·757	9·613
30	5·098	11·007
35	6·226	12·452
40	6·639	14·817
45	8·759	19·515
50	11·789	23·888
55	13·717	25·116
60	13·212	28·455
65	17·563	33·530
70	19·368	33·297
75	17·275	34·515
80	20·840	34·339
85	17·052	11·014
90	(−)7·280	

Age when Assured.	50	55
15	[illegible]	63·641
20	62·334	69·629
25	68·298	73·775
30	72·465	78·203
35	77·082	80·949
40	79·684	78·205
45	76·655	

Table No. 20.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to a Hypothetical Table formed from Carlisle 3 per Cent Premiums, with a loading of 13 per Cent.*

Age when Assured.	5	10
15	4·071	8·762
20	4·889	10·117
25	5·496	11·021
30	5·846	12·506
35	7·074	14·021
40	7·476	16·502
45	9·755	21·447
50	12·956	25·902
55	14·873	26·933
60	14·167	30·130
65	18·597	35·104
70	20·277	34·580
75	17·940	35·559
80	21·470	35·196
85	17·478	11·310
90	(−)7·475	

Age when Assured.	50	55
15	59·937	67·388
20	66·004	72·697
25	71·504	76·616
30	75·256	80·569
35	79·362	82·969
40	81·673	80·303
45	78·711	

Table No. 21.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to a Hypothetical Table formed from Carlisle 3 per Cent Premiums, with a loading of 25 per Cent.*

s when sured.	5	10	15
15	4·333	9·297	14·57
20	5·189	10·703	16·20
25	5·816	11·620	18·18
30	6·162	13·133	19·92
35	7·428	14·670	23·34
40	7·822	17·188	28·30
45	10·160	22·219	34·13
50	13·422	26·691	37·34
55	15·326	27·635	41·37
60	14·537	30·766	45·04
65	18·990	35·691	47·38
70	20·617	35·053	49·14
75	18·186	35·940	47·28
80	21·701	35·507	30·64
85	17·633	11·417	
90	(−)7·546		

s when sured.	50	55	60
15	61·498	[illegible]	75·24
20	67·397	74·118	78·82
25	72·702	77·666	82·51
30	76·287	81·433	84·70
35	80·214	83·703	82·47
40	82·395	81·066	
45	79·460		

Table No. 22.—*Showing the Values of Policies of £100 taken out at various ages, and in force for various terms, according to a Table formed from Average Premiums, reckoning Interest at 3 per Cent.*

TABLE NO. 23.—*Showing the Values of Policies, according to the Carlisle Table and 3 per Cent Interest, when the gross Premiums are valued; the premiums being formed from the Carlisle 3 per Cent, with a loading of 25 per Cent.*

Age when Assured.	5	10
15	− 3·769	·926
20	− 3·561	1·751
25	− 3·627	2·034
30	− 4·016	2·889
35	− 3·488	3·814
40	− 3·905	5·708
45	− 2·294	10·373
50	+ 0·237	14·500
55	+ 1·245	14·769
60	− 0·657	17·554
65	+ 3·530	22·663
70	+ 4·535	21·350
75	+ 0·924	21·902
80	+ 4·542	21·020
85	− 0·869	− 8·308
90	−31·493	

Age when Assured.	50	55
15	54·651	62·857
20	61·174	68·875
25	67·172	72·954
30	71·284	77·364
35	75·877	80·042
40	78·440	76·850
45	74·886	

TABLE No. 24.—*Showing the Values of Policies according to the 6th Method, calculated by the Carlisle Table, reckoning Interest at 3 per Cent; the premiums charged being formed from the Carlisle 3 per Cent, with a loading of 25 per Cent, and reduced 50 per Cent after 5 years.*

Age when Assured.	Number of years elapsed.								
	5	10	15	20	25	30	35	40	45
15	15·066	18·907	23·068	27·270	32·098	36·889	42·728	49·820	56·994
20	16·663	20·940	25·258	30·219	35·143	41·143	48·431	55·804	61·854
25	18·251	22·716	27·846	32·937	39·142	46·678	54·301	60·557	67·694
30	19·691	25·022	30·313	36·760	44·591	52·512	59·014	66·430	73·088
35	21·834	27·349	34·070	42·234	50·493	57·271	65·002	71·944	76·886
40	23·490	30·568	39·166	47·863	55·001	63·144	70·454	75·658	80·812
45	26·567	35·660	44·859	52·408	61·020	68·751	74·255	79·706	83·210
50	30·592	40·515	48·660	57·949	66·290	72·227	78·107	81·887	80·551
55	33·888	42·940	53·264	62·534	69·133	75·668	79·869	78·384	
60	35·227	46·947	57·469	64·960	72·379	77·148	75·462		
65	39·935	51·848	60·329	68·729	74·127	72·219			
70	42·784	52·862	62·842	69·257	66·989				
75	42·649	54·792	62·596	59·837					
80	46·112	55·414	52·126						
85	44·470	40·374							
90	28·782								

Age when Assured.	Number of years elapsed.								
	50	55	60	65	70	75	80		
15	62·881	69·598	75·628	79·921	84·172	86·905	85·939		
20	68·757	74·954	79·366	83·734	86·542	85·550			
25	74·102	78·664	83·181	86·085	85·058				
30	77·829	82·523	85·540	84·473					
35	81·779	84·925	83·813						
40	84·124	82·953							
45	81·971								

TABLE No. 25.—*Showing the Present Value of Premiums on Policies taken out at various ages and continued in force for various terms, according to the Carlisle Table and 3 per Cent Interest; the premiums being formed from the Carlisle 3 per Cent, with a loading of 25 per Cent.*

Age when Assured.	Number of years elapsed.								
	5	10	15	20	25	30	35	40	45
15	37·670	35·962	34·121	32·257	30·116	27·991	25·401	22·256	19·074
20	40·449	38·378	36·281	33·873	31·483	28·571	25·033	21·454	18·515
25	43·756	41·365	38·619	35·895	32·574	28·540	24·460	21·109	17·291
30	47·415	44·267	41·144	37·338	32·714	28·037	24·196	19·819	15·889
35	50·644	47·071	42·717	37·427	32·076	27·682	22·675	18·178	14·976
40	54·790	49·721	43·564	37·335	32·221	26·392	21·158	17·431	13·740
45	57·723	50·575	43·344	37·407	30·640	24·563	20·237	15·952	13·198
50	60·711	52·031	44·904	36·781	29·486	24·293	19·149	15·843	17·012
55	65·286	56·343	46·151	36·998	30·481	24·027	19·880	21·345	
60	71·769	58·786	47·127	38·827	30·605	25·322	27·189		
65	72·810	58·370	48·089	37·906	31·363	33·675			
70	76·498	63·024	49·679	41·104	44·135				
75	83·450	65·780	54·426	58·439					
80	83·140	68·789	73·861						
85	90·678	97·365							
90	120·550								

Age when Assured.	Number of years elapsed.								
	50	55	60	65	70	75	80		
15	16·461	13·483	10·809	8·905	7·020	5·808	6·236		
20	15·166	12·158	10·016	7·896	6·533	7·014			
25	13·861	11·420	9·002	7·448	7·997				
30	13·090	10·318	8·537	9·167					
35	11·805	9·767	10·487						
40	11·369	12·207							
45	14·171								

TABLE No. 26.—*Showing the present value of* £100 *payable at the end of the year in which an assigned life may fail, according to the Carlisle Table of Mortality, reckoning Interest at* 3 *per Cent.*

Age (x).	Value $\pi_x(1+a_x)$.	Age (x).	Value $\pi_x(1+a_x)$.
20	33·901	60	66·531
25	36·888*	65	71·112
30	40·129	70	76·340
35	43·399	75	81·033
40	47·156	80	84·374
45	50·885	85	87·682
50	55·429	90	89·809
55	60·948	95	89·057

A short perusal of these Tables will soon lead us to expect that in practice, where assurances to the amount of millions exist, the variations in the reserves made by the use of different data would be very great indeed. I will endeavour to show, as clearly as possible, the different results which a practical employment of the various data and methods used in the valuation of the liabilities of an Office might be expected to produce. To do this it will be necessary first to form some definite idea of the total amounts assured at different ages in an ordinary Office, and of the proportions remaining in force at the end of various periods of its existence.

There are many difficulties in the way of arriving at any satisfactory result to suit all cases; in the first place, there are circumstances which tend, though practically not to the extent which might be expected, to vary the proportion of Assurances at different ages in different Offices; thus, we might expect to find that an Office charging very low premiums at the younger ages would have a greater proportion of young lives than one charging very high premiums at the same ages; and *vice versâ*, that an Office charging very low premiums at the older ages, would have a greater number assuring at those ages than one charging very high premiums. Again, the mode of distributing the profits is calculated to have a similar effect, but in a less degree; for instance, an Office using a method which gives an advantage to the young lives over the old, would be likely to have a greater proportion of young lives; and where the advantage is on the side of the old lives, an increase in the number assuring at those ages might be expected.

The means at our disposal to guide us to a true result are very scanty. I only know of two instances where the amounts assured existing at different ages have been published. The first is a paper of the liabilities of the Equitable Society, given in by Mr. A. Morgan

* This value is incorrectly printed in Jones, p. 538, as 36·808.

to the Parliamentary Committee on Joint Stock Companies, and which was printed on page 57 of their Report;* the other is that of an Abstract of the Policies in force in the year 1862 in the late National Mercantile Office, and which was printed in their Report for that year. (See *Insurance Record,* vol. i. p. 187.)

There is a great difference between these Tables, even beyond what might be expected from the difference in the ages of the two Offices at the time, and I think there are objections to a strict adherence to either of them. In the case of the Equitable Society, the circumstance that the amount of New Assurances had been gradually falling off for some time would alone cause an increase in the Assurances in force at the older ages compared with an Office transacting a stationary or increasing new business; and this difference would be increased by a smaller proportion entering at the younger ages on account of the high premiums charged. In the case of the National Mercantile, its business had been very unsatisfactory for some years, and cannot therefore be safely taken as a standard to go upon.

When we come to consider the proportions remaining in force at the end of various periods of the existence of an Office, we find that the chances of arriving at any satisfactory conclusion become still more remote. Thus, I take as example the experience of three Offices, all of them of the highest respectability. The first, which had been established 20 years, had 30 per cent of the policies issued become void during that period; in the second, which had been established 30 years, 25 per cent had become void; while in the third, established 45 years, no less than 65 per cent of the policies issued had become void.

Under these circumstances I considered it advisable to construct a Table for the purposes required, entirely independent of such unsatisfactory data and conflicting statements. In the construction of this Table, which I give below, and which I hope will be considered satisfactory, I have endeavoured to steer a medium course; but if I have erred, it has been in making the proportions which become void larger than I probably ought to have done; if it be so, I consider that the error is on the right side, for then it will represent the condition of an Office with a more or less increasing business.

For the sake of simplicity, I have supposed that all who enter do so at the beginning of the year and at intervals of 5 years, thus, I suppose £1,300,000 to be assured by persons entering at the commencement of the first year, and that the same number of persons assuring for the same amount enter at the end of 5 years, in

* See also Registrar-General's 12th Report, Appendix, p. 61.

which time the first amount has been reduced by claims, lapses, surrenders, &c., to £874,550, which amount at the end of another 5 years is still further reduced by similar causes to £707,350, and so on.

TABLE NO. 27—*Showing the assumed amounts of Policies taken out at various ages and remaining in force at the end of stated periods, for the purpose of exemplifying the different results obtained by the use of different data in the valuation of the liabilities of an Office.*

Age at entry.	Number of years in force.						
	0	5	10	15	20	25	30
15	500	300	250	200	150	150	100
20	9,000	6,000	5,000	4,000	3,000	2,500	2,000
25	40,000	30,000	20,000	15,000	14,000	11,000	9,000
30	150,000	100,000	80,000	60,000	50,000	40,000	30,000
35	250,000	170,000	140,000	110,000	90,000	70,000	50,000
40	300,000	200,000	160,000	130,000	100,000	90,000	60,000
45	250,000	170,000	140,000	110,000	80,000	60,000	30,000
50	180,000	120,000	100,000	70,000	50,000	30,000	15,000
55	90,000	60,000	50,000	30,000	20,000	9,000	4,000
60	25,000	15,000	10,000	7,000	4,000	1,500	500
65	5,000	3,000	2,000	900	400	100	100
70	500	250	100	100			
Totals	1,300,000	874,550	707,350	537,200	411,550	314,250	200,700

Age at entry.	Number of years in force.					
	35	40	45	50	55	60
15	100	100	..	..	..	..
20	1,500	1,000	900	600	400	200
25	7,000	5,000	3,000	2,000	1,000	400
30	25,000	15,000	10,000	5,000	2,000	600
35	30,000	20,000	12,000	5,000	1,500	300
40	35,000	18,000	7,000	2,000	400	..
45	15,000	7,000	2,000	400	..	
50	6,000	2,000	400	..		
55	1,000	200	..			
60	100	..				
Totals	120,700	68,300	35,300	15,000	5,300	1,500

It is upon this basis that I have constructed the following Table (No. 28), which exhibits the reserve made at the end of every quinquennial period by each Table of Mortality, each rate of Interest, and each method of Valuation which I have previously used. I have also added to each Value the increase or decrease per cent on the corresponding Value by the Carlisle 3 per cent Table.

This Table I hope will be considered of some value, since by means of it we shall be able to ascertain approximately what the result of a valuation would have been if different data to those employed had been used.

TABLE No. 28.—*Showing the Reserve made by each Table of Mortality, each Rate of Interest, and each Method of Valuation; and a comparison of each with the Carlisle 3 per Cent value.*

	Age of Office.							
	5 years.		10 years.		15 years.		20 years.	
rest	Reserve.	Per Centage greater (+) or less (−) than Carlisle 3 per Cent.	Reserve.	Per Centage greater (+) or less (−) than Carlisle 3 per Cent.	Reserve.	Per Centage greater (+) or less (−) than Carlisle 3 per Cent.	Reserve.	Per Centage greater (+) or less (−) than Carlisle 3 per Cent.
....	74493		200261		346380		496512	
....	69930	− 6·125	188948	− 5·648	328177	− 5·254	472144	− 4·908
....	65695	− 11·811	178387	− 10·921	311105	− 10·182	449199	− 9·529
....	81262	+ 9·087	216660	+ 8·189	372303	+ 7·484	530653	+ 6·877
....	76370	+ 2·520	[illegible]	+ 2·333	353678	+ 2·107	[illegible]	+ 1·893
....	72007	− 3·337	194140	− 3·056	336347	− 2·896	482732	− 2·775
....	78518	+ 5·403	209640	+ 4·684	360584	+ 4·100	[illegible]	+ 3·647
....	74029	− 0·623	198562	− 0·849	342824	− 1·028	[illegible]	− 1·130
....	69854	− 6·227	188200	− 6·022	326133	− 5·846	468521	− 5·637
....	78425	+ 5·278	[illegible]	+ 4·769	361296	+ 4·307	515961	+ 3·915
....	69667	− 6·478	188135	− 6·052	326472	− 5·748	469386	− 5·462
....	71741	− 3·694	191222	− 4·514	328999	− 5·018	470165	− 5·305
....	67383	− 9·545	180375	− 9·927	311460	− 10·081	446534	− 10·066
....	63351	− 14·957	170286	− 14·966	[illegible]	− 14·813	424357	− 14·531
....	78261	+ 5·058	209747	+ 4·739	361736	+ 4·434	517280	+ 4·183
....	68661	− 7·829	186290	− 6·976	323873	− 6·499	465435	− 6·260
....	68108	− 8·573	181980	− 9·128	314008	− 9·345	450449	− 9·277
....	60637	− 18·602	163228	− 18·491	283466	− 18·162	409049	− 17·615
....	70474	− 5·395	190115	− 5·063	329725	− 4·807	474000	− 4·534
).........	78565	+ 5·466	*210479	+ 5·103	362961	+ 4·787	518854	+ 4·499
)	81879	+ 9·914	218747	+ 9·233	376315	+ 8·641	536770	+ 8·109
le No. 23)..	(−)21918	− 129·423	33999	− 83·023	133649	− 61·415	253145	− 49·015
Table No. 24)	219107	+ 194·131	449657	+ 124·537	665486	+ 92·126	861581	+ 73·528
isle 3 per Ct.)	460133		865315		1197324		1470018	
$_x = 1{\cdot}25\pi_x$)	482051		831316		1063675		1216873	

TABLE No. 28 (continued).

Age of Office.						
…s.	30 [illegible]		35 years.		40 years.	
…er Centage …reater (+) …r less (−) …an Carlisle … per Cent.	Reserve.	Per Centage greater (+) or less (−) than Carlisle 3 per Cent.	Reserve.	Per Centage greater (+) or less (−) than Carlisle 3 per Cent.	Reserve.	Per Centage greater (+) or less (−) than Carlisle 3 per Cent.
	744746		816895		862122	
− 4·613	711964	− 4·401	782135	− 4·255	826303	− 4·155
− 8·971	680912	− 8·572	749156	− 8·292	792284	− 8·101
+ 6·433	790624	+ 6·160	865810	+ 5·987	912902	+ 5·890
+ 1·774	757635	+ 1·730	830911	+ 1·715	876995	+ 1·725
− 2·611	726499	− 2·450	797912	− 2·325	[illegible]	− 2·218
+ 3·349	768430	+ 3·180	842006	+ 3·074	888048	+ 3·008
− 1·132	736632	− 1·090	808322	− 1·050	853358	− 1·016
− 5·375	706450	− 5·143	776299	− 4·970	820347	− 4·845
+ 3·646	770607	+ 3·470	[illegible]	+ 3·353	890376	+ 3·278
− 5·164	708037	− 4·929	777967	− 4·766	822046	− 4·648
− 5·385	705426	− 5·280	773038	− 5·124	819382	− 4·958
− 9·886	673289	− 9·595	740858	− 9·308	784106	− 9·049
−14·126	642930	−13·672	708514	−13·268	750690	−12·925
+ 4·017	773803	+ 3·901	848035	+ 3·812	894462	+ 3·751
− 6·047	700858	− 5·893	769591	− 5·791	812948	− 5·703
− 9·026	680181	− 8·670	748767	− 8·340	792601	− 8·064
−16·953	623412	−16·293	688286	−15·744	730117	−15·311
− 4·295	714077	− 4·118	784289	− 3·992	828471	− 3·903
+ 4·343	775605	+ 4·142	849609	+ 4·004	895830	+ 3·910
+ 7·645	799122	+ 7·300	874564	+ 7·058	921549	+ 6·893
−41·230	470818	−36·782	537610	−34·189	580335	−32·685
+61·850	1155658	+55·174	1235840	+51·284	1284822	+49·030
…22	1840498		1934070		1989309	
…30	1369679		1396460		1408974	

TABLE No. 28 (continued).

Age of Office.							
45 years.		50 years.		55 years.		60 years.	
...rve.	Per Centage greater (+) or less (–) than Carlisle 3 per Cent.	Reserve.	Per Centage greater (+) or less (–) than Carlisle 3 per Cent.	Reserve.	Per Centage greater (+) or less (–) than Carlisle 3 per Cent.	Reserve.	Per Centage greater (+) or less (–) than Carlisle 3 per Cent.
'279		898595		902769		903983	
944	– 4·096	862056	– 4·066	866162	– 4·055	867359	– 4·051
416	– 7·986	827326	– 7·931	831366	– 7·909	832545	– 7·902
119	+ 5·842	[illegible]	+ 5·826	955350	+ 5·824	956645	+ 5·826
731	+ 1·741	914383	+ 1·756	918721	+ 1·767	920001	+ 1·772
264	– 2·144	879732	– 2·100	[illegible]	– 2·078	885279	– 2·069
632	+ 2·970	925138	+ 2·954	929387	+ 2·948	930630	+ 2·948
455	– 0·994	889769	– 0·983	893956	– 0·976	895184	– 0·973
.963	– 4·770	856088	– 4·731	860213	– 4·714	861424	– 4·708
988	+ 3·236	927512	+ 3·217	931771	+ 3·212	[illegible]	+ 3·213
684	– 4·576	857826	– 4·537	861960	– 4·521	863176	– 4·513
.412	– 4·832	855842	– 4·758	860117	– 4·724	861385	– 4·712
608	– 8·866	819835	– 8·765	824047	– 8·720	825299	– 8·704
673	–12·692	785699	–12·564	789848	–12·508	791084	–12·489
262	+ 3·717	931876	+ 3·704	936171	+ 3·700	937431	+ 3·700
214	– 5·643	848214	– 5·608	852294	– 5·592	853489	– 5·585
378	– 7·878	828693	– 7·780	832930	– 7·736	834189	– 7·720
925	–15·029	764866	–14·882	768984	–14·820	770214	–14·798
118	– 3·850	864234	– 3·825	868341	– 3·814	869538	– 3·810
468	+ 3·853	932973	+ 3·825	937208	+ 3·815	938489	+ 3·812
555	+ 6·794	959204	+ 6·744	963486	+ 6·726	964729	+ 6·720
418	–31·880	615355	–31·521	619415	–31·387	620602	–31·348
588	+47·821	1323478	+47·282	1327816	+47·082	1329072	+47·025
2018758 1414340		2031592 1416237		2036216 1416801		2037542 1416940	

By the last two lines which I have added to the Table, it will be easy to ascertain what result any particular deduction made in the use of the 5th method would produce. The amount of reserve made by the 6th method may vary from the Values in the last line but one (the present value of the sums assured) to those in the last line but four (the values by Method No. 4), which I think will fully justify the name I have given to it—that of the Elastic Method.

Here we can distinctly observe the variations in the results produced by the valuation of the Liabilities of an Office upon different data. For example, we see that the reserve made by an Equitable 3 per cent valuation is almost the same as a valuation by either of the English Life Tables, reckoning interest at 4 per cent, and is much less than that made by the Experience Table and 4 per cent interest. In the earlier years, it gives a slightly larger reserve than the Carlisle 3½, but after 15 or 20 years it becomes smaller. Again, a Carlisle 3 per cent valuation gives a slightly larger reserve than one made by either of the English Life Tables at 3½ per cent, but shows a considerable difference the other way when compared with the results of the Experience Table at 3½ per cent. It is not necessary to multiply examples, as the differences can be distinctly seen by a reference to the Table.

These results are very striking, and cannot but lead us seriously to consider, whether the reserves made by some Offices may not be too small, notwithstanding that a low rate of Interest has been adopted. For instance, suppose that an Office valuing by the Equitable Table were to experience a mortality corresponding to that exhibited by the Combined Experience Table; such an Office, although it would take credit for having made its calculations at a rate so low as 3 per cent, would have a sum in reserve so small as to require accumulating at a rate of interest over 4 per cent, in order to meet all its claims and maintain a solvent condition. But if the Office were to use 4 per cent in its valuations, instead of 3 per cent, it would have to make nearly 5½ per cent on its investments before it could be said to be perfectly solvent. Thus, at the end of 30 years, the reserve by the Experience Table at 3½ per cent is 4·17 per cent less than at 3 per cent, and at 4 per cent it is 4·11 per cent less than at 3½ per cent; so that, assuming similar proportions to subsist between higher rates of interest, the reserves at 4½ per cent, 5 per cent and 5½ per cent by the Experience Table would be about £697,000, £669,000 and £642,000 respectively, the last being nearly the same as the

Equitable 4 per cent. This, it may be said, is taking an extreme view of the case, but then it is only by such examples that we can give force to an argument.

We can clearly observe that the difference between a 3 per cent and a 4 per cent valuation is very large, and that this difference gradually diminishes in proportion as the Office grows older; but it is neither so large as has been stated by some actuaries using 3 per cent, nor so small as those using 4 per cent have sometimes declared it to be. In the Investigation Report of an Office, transacting an extensive business, there appear statements of such an astounding character that I cannot refrain from quoting a passage or two from it, if only to show how the public may be misled upon this subject —whether intentionally, or not. After stating that if, in the fifteenth year of its existence, the valuations had been made reckoning interest at 4 per cent instead of 3 per cent, the estimated amount of Liabilities would have been £68,000 instead of £147,000; and also, that if the gross premiums had been valued at 4 per cent, then no reserve would have been required—it goes on to state that "The difference in the valuation at different rates "of interest, is far greater on *young* lives and on Policies *recently* "effected than on those in existence for a long period and at older "ages, where the remaining expectation of Life over which the "advantage of an increased rate is to run is obviously so much "less. The effect is curiously exemplified in the case of Policies "only three or four years in existence.

	At 3 per cent.	At 4 per cent, estimated about
"Thus, the valuation of Participating "Policies on single lives effected (during "the first 10 years) is	£79,279	£54,700
"Whilst a like valuation of Policies "effected (during the next 5 years) is	46,496	160!"

The *note of admiration* is not mine, but is in the Report.

Now, by reference to the last Table, we find that the difference between the values of the Liabilities when interest is reckoned at 3 per cent and 4 per cent, is, after 15 years, between 10 per cent and 11 per cent; and therefore, in the above Report, instead of estimating the Liabilities, when calculated at 4 per cent, at £68,000, it ought to have been about £131,000—a very considerable difference indeed. Again, the difference between the values of the Liabilities at the end of 10 years, when interest is reckoned at 3 per cent and 4 per cent, is about 11 per cent; and therefore the estimated value of the Policies effected during the first 10 years, reckoning interest at 4 per cent, should have been about £70,000

and not £54,700 as stated: likewise, the value of the Policies effected during the first 5 years, reckoning interest at 4 per cent, should have been estimated at about 12 per cent less than the 3 per cent valuation, which would have made it £40,900 and not £160!

I have however found that results similar to those stated in the Report, would be produced by valuing 3 per cent premiums by 4 per cent annuities; and this, I presume, is what is to be understood by a valuation made at 4 per cent. With this explanation I leave the reader to judge of the fairness and, indeed, the truth of the statements.

As it would be interesting to know the exact difference between the values of the Policies as they grow older when calculated at 3, 3½, and 4 per cent, I have made the following Table, which shows the difference per cent between the values, at the different rates, of the Policies existing in an Office which have been in force for different periods. Thus, the difference in the values at 3 and 4 per cent of all those which have been in force for only 5 years, according to Table No. 27, is 11·811 per cent by the Carlisle Table, and 11·034 per cent by the English Life Table No. 3; and the difference between the values of all those that have been in force for exactly 40 years is, under similar circumstances, 4·641 per cent by the Carlisle Table, and 4·329 by the English Life No. 3.

Table No. 29.—Showing the difference per Cent between the values of all the Policies of different ages, when different rates of Interest are used.

Age of Policies.	According to Carlisle Table.			According to English Life Table No. 3.		
	Difference between 3 per Cent and 3½ per Cent.	Difference between 3½ per Cent and 4 per Cent.	Difference between 3 per Cent and 4 per Cent.	Difference between 3 per Cent and 3½ per Cent.	Difference between 3½ per Cent and 4 per Cent.	Difference between 3 per Cent and 4 per Cent.
5	6·125	6·056	11·811	5·717	5·640	11·034
10	5·367	5·317	10·399	5·025	4·968	9·744
15	4·715	4·676	9·171	4·427	4·386	8·620
20	4·107	4·080	8·019	3·868	3·842	7·561
25	3·582	3·570	7·025	3·370	3·359	6·615
30	3·132	3·128	6·162	2·934	2·932	5·781
35	2·742	2·746	5·412	2·563	2·568	5·066
40	2·342	2·355	4·641	2·185	2·192	4·329
45	2·051	2·066	4·074	1·904	1·917	3·784
50	1·803	1·818	3·588	1·669	1·679	3·320
55	1·606	1·534	3·115	1·459	1·481	2·918
60	1·459	1·526	2·963	1·286	1·303	2·572

The difference grows gradually less as the policies grow older; and therefore, if the proportion of policies becoming void is prac-

tically less than I have supposed in Table No. 27, the difference in the proportion of the amounts reserved, as shown in Table No. 28, will be slightly less as the Office increases in age.

Having ascertained the amounts reserved at successive valuations, we are now able to consider in what manner the divisible surplus, present and future, will be influenced by the employment of any particular data or method.

Let us suppose there are two Offices, A and B, each transacting a business exactly corresponding to Table No. 27; but that A values its liabilities by the Carlisle Table, and B by the Experience Table—both using 3 per cent rate of interest. Let us also suppose the rate of interest realized to be 4 per cent; and that the liabilities are valued every five years. At the first valuation the reserve made by A will be £74,493, while B will make a reserve of £81,262, being larger than that made by A by £6,769; which amount will be returned by A, as realized profit, over and above that which B will distribute. At the next valuation, the amount reserved by A will be £16,399 less than will be made by B, but then B will have in hand funds larger than A's by a sum equal to the difference in the reserve made on the last occasion (£6,769), accumulated at compound interest, during the interval, at the rate realized, that is by £8,235; and, consequently, A will distribute as realized profit at the second valuation £8,164 (=£16,399—£8,235) more than B. At the end of the third quinquennial period, B will have in hand funds greater than A's, to the extent of the difference in the reserve made at the second valuation (£16,399) accumulated at 4 per cent compound interest during the interval; which amount will be £19,952. This time A will make a reserve of £25,923 less than B, and will, therefore, divide the sum of £5,971 only more than B. By continuing this process of reasoning, we shall find that, at the next valuation, this difference will be considerably less; while, at the fifth investigation, B will actually have a divisible surplus of £467 more than A, and at the same time possess a larger Reserve fund by £41,070, or 6·43 per cent on the whole amount reserved by A.

At the sixth period, the difference in the sums to be divided will be £4,090 in favour of B; while, at the eighth valuation, and the 40th year of the existence of the Offices, B will distribute, as realized profit, an amount greater than A by £8,731, a larger sum than was ever distributed by A over B during the first years of their existence, when A had the advantage; and yet, notwithstanding this difference in the divisible surplus, B will still make a

reserve much greater than A—for while B will value its liabilities at £912,902, A will put their value down as £862,122, a difference of £50,780, or 5·89 per cent on the whole reserve made by A.

This is a most important fact, and worthy of the greatest consideration from all the members of the Actuarial profession: for here we distinctly see that an Office in which the liabilities are valued by a Table giving a large reserve will, after 25 or 30 years, make a larger return of bonus than if its liabilities had been valued by a Table giving a smaller reserve; while, during the first 15 or 20 years, the difference in the bonus is so small, that it might be really considered insignificant when compared with the increased security of the Office. And again, during the first years of the existence of an Office, the effect of selection is such, that it could really afford, if managed with proper care and economy, to make a large reserve, especially if it be desired to have a gradually increasing bonus in future years.

Besides all this, the larger reserve gives a greater guarantee against injury from possible large fluctuations in the mortality of an Office; and should it ever be considered desirable to alter the Table by which the valuations are made, it would be extremely difficult for an Office using a Table which gives a small value, to adopt one making a larger value. For instance, suppose that A, in consequence of the mortality prevailing amongst the assured being different to what was anticipated, were at any time—say at the end of 25 years—to alter the Table by which it valued, from the "Carlisle" to the "Experience," the difference between A and B would be as follows:—B would have in hand a sum equal to the difference between the reserves at the last valuation accumulated at the interest realized—that is, £41,537, which will represent the difference in the divisible surplus returned by B over A; since, on this occasion, they would both make the same reserve. Such a difference would have a considerable effect upon the amount of A's bonuses.

The above amount, £41,537, will also represent the difference in the divisible surplus in favour of B, should that Office, at the fifth valuation, alter its Table from the "Experience" to the "Carlisle"; A of course still continuing to value by that Table. We cannot but conclude after this that the advantage gained by the use of a Table making a large Reserve, far surpasses any advantage which the use of a Table making a smaller reserve would give in an increased surplus during the first few years of the existence of the Office.

In the above remarks I have considered that the surplus has been returned to the Assured in cash at the time of valuation; but should it be the practice of the Office to add an equivalent reversionary sum to the amount assured, it will make only a slight difference in the results there stated. Thus, the difference in the surplus at the first valuation will be the same,—namely £6,769; and A will consequently give as bonus a reversionary sum larger than B, equivalent to this difference, and which we will suppose to be calculated at 3 per cent. That amount, therefore, will be such that, £6,769, when accumulated at 3 per cent interest, will exactly meet all those surplus bonuses as they become claims; so that, at the next valuation, the amount to be reserved by A over B to meet these surplus bonuses will be £6,769, less the claims paid during the interval, accumulated at 3 per cent compound interest; while, at the same time, the funds of A will be less than those of B by the amount of surplus bonuses paid accumulated at the interest realized; consequently, the difference in the divisible surplus in favour of A, in the second investigation, will be £16,399, less (£6,769 accumulated at 3 per cent, *i.e.*) £7,847, which is £8,552. This amount will be slightly less if the interest realized has been more than 3 per cent, or if A has paid any of the bonuses in cash. The same reasoning will apply to every successive valuation, so that, if we assume the rate of interest realized to be 3 per cent, instead of 4 per cent, we shall ascertain very nearly what the difference between the surplus divided by A and B will be, when it is returned to the assured in the shape of an equivalent reversionary bonus. Under these circumstances we shall find that, at the sixth valuation, B will divide a larger surplus than A, as will be distinctly seen by reference to the following Table, which plainly exhibits the different results produced by the use of the two Tables when the interest realized is either 3, 4, 4½ or 5 per cent.

I have selected the Carlisle and Experience Tables because they are more frequently used than others; but, by the aid of Table No. 28, it could be easily shown that, if any other Tables of Mortality had been used, they would have produced similar results; the advantage in favour of the Table giving the larger value almost invariably appearing at or before the 30th year of the existence of the Office.

Nor are these advantages alone on the side of the Table, but they are also in favour of a small rate of interest, for when we come to consider the difference produced in the surplus from the employment of 3 per cent and 4 per cent, we find that the

advantage in favour of a 3 per cent valuation begins to show itself in about 25 years; and that, after that time, the divisible surplus in favour of the smaller rate increases rapidly.

TABLE No. 30.—*Showing the difference in the Reserve and Divisible Surplus between an Office* (A) *valuing by Carlisle* 3 *per Cent and an Office* (B) *valuing by Experience* 3 *per Cent.*

Age of Office (years).	Reserve made by (B) over (A).	Ditto, accumulated for five years, at				Surplus divided by (A) over (B) when Interest realized is				Surplus divided by (B) over (A) when Interest realized is			
		3 per Cent.	4 per Cent.	4½ per Cent.	5 per Cent.	3 per Cent.	4 per Cent.	4½ per Cent.	5 per Cent.	3 per Cent.	4 per Cent.	4½ per Cent.	5 per Cent.
5	6769	7847	8235	8434	8638	6769	6769	6769	6769				
10	16399	19011	19952	20436	20929	8552	8164	7965	7761				
15	25923	30050	31539	32305	33084	6912	5971	5487	4994				
20	34141	39578	41537	42545	43573	4091	2602	1836	1057		...	....	....
25	41070	47611	49968	51182	52417	1492	....	....	....	....	467	1475	2503
30	45878	53184	55817	57172	58553	....				1733	4090	5304	6539
35	48915	56706	59511	60955	62428					4269	6902	8257	9638
40	50780	58867	61781	63280	64810					5926	8731	10175	11648
45	51840	60098	63071	64602	66163					7027	9941	11440	12970
50	52360	60699	63704	65249	66826					7738	10711	12242	13803
55	52581	60955	63972	65524	67108					8118	11123	12668	14245
60	52662									8293	11310	12862	14446

To show the exact influence of the different rates at successive periods, I have constructed the two following Tables; in the first of which, the 4 per cent column will show the effect when the bonus is returned as an equivalent reversionary sum added to the policy.

TABLE No. 31.—*Showing the difference in the Reserve and Divisible Surplus when the liabilities of an Office are valued by Carlisle* 3 *per Cent and Carlisle* 4 *per Cent; and when the Interest realized is* 4 *per Cent and* 4½ *per Cent.*

Age of Office (years).	Reserve made by Carlisle 3 per Cent over Carlisle 4 per Cent valuation.	Ditto, accumulated for five years at		Divisible Surplus in favour of 4 per Cent Valuation when Interest realized is		Divisible Surplus in favour of 3 per Cent Valuation when Interest realized is	
		4 per Cent.	4½ per Cent.	4 per Cent.	4½ per Cent.	4 per Cent.	4½ per Cent.
5	8798	10704	10964	8798	8798		
10	21874	26613	27259	11170	10910		
15	35275	42917	43959	8662	8016		
20	47313	57563	58961	4396	3354	....	....
25	57286	69696	71388	....	....	277	1675
30	63834	77673	79550			5862	7554
35	67739	82414	84415			9934	11811
40	69838	84968	87032			12576	14577
45	70863	86216	88309			14105	16169
50	71269	86710	88814			14947	17040
55	71400	86869	88978			15310	17414
60	71436					15433	17542

Table No. 32.—*Showing the difference in Reserve and Divisible Surplus between Carlisle* 3 *per Cent and Carlisle* $3\frac{1}{2}$ *per Cent Valuations, and also between Carlisle* $3\frac{1}{2}$ *per Cent and Carlisle* 4 *per Cent Valuations—when Interest realized is* $4\frac{1}{2}$ *per Cent.*

Age of Office (years).	Reserve made by Carlisle 3 per Cent over Carlisle 3½ per Cent.	Surplus in favour of Carlisle 3½ per Cent.	Surplus in favour of Carlisle 3 per cent.	Reserve made by Carlisle 3½ per Cent over Carlisle 4 per Cent.	Surplus in favour of Carlisle 4 per Cent.	Surplus in favour of Carlisle 3½ per Cent.
5	4563	4563		4235	4235	
10	11313	5627		10561	5283	
15	18203	4105		17072	3911	
20	24368	1683	....	22945	1671	....
25	29454	....	913	27832	....	762
30	32782		3923	31052		3631
35	34760		6091	32979		5720
40	35819		7498	34019		7079
45	36335		8302	34528		7867
50	36539		8741	34730		8299
55	36607		8927	34793		8487
60	36624		8995	34812		8547

Those Offices which value by a Hypothetical Table (Method No. 2), will, where the Table gives a larger value than by the pure method, give a smaller bonus for about the first 25 years, after which they will distribute a larger surplus than if the liabilities were valued by the net method.

An Office valuing by the fourth method, notwithstanding the false position it places itself in by taking credit for the punctual payment in future of all the premiums in full, and supposing that the Office can be carried on without expenses, will divide a larger surplus at the first three valuations only; at the fourth it will be about equal; and, after that, much smaller than by the net method, diminishing with great rapidity as the Office grows older.

Method No. 6 is, as I have before stated, so elastic, and is confined to Offices which divide the surplus in a manner peculiar to themselves, that it is almost impossible to make a fair comparison with any other method. If they continue to manage their business in future with the economy which has generally characterized their past career, there is no reason to doubt that they will continue to make very large abatements in their premiums; and so long as they do that, in consequence of the peculiarity of their system of valuation, they will always have a very large reserve.

It is impossible to say what the divisible surplus by the third method will be, as it depends so much on the management and experience of the Office; but we can tell what it ought to be, when

we have fixed upon a true Table and rate of interest. As I have previously stated, a large Office has lately declared that a valuation by the Northampton 4 per cent, reserving one-third of the resulting surplus, gives a result corresponding very nearly to the Carlisle 3 per cent. Their surplus must have been, therefore, three times the difference between the reserves made by a Carlisle 3 per cent and Northampton 4 per cent valuations. When such is the case, the same remarks that apply to the Carlisle 3 per cent net method will also apply to this.

It cannot fail to have been noticed by those readers who are at all acquainted with the Tables of mortality which I have used in this Essay, that those Tables which present a high rate of mortality give, in some instances, much smaller values than Tables exhibiting a lower rate; to exemplify which I have only to mention the Northampton and Experience Tables—the first presenting a rate of mortality for many years more than double that of the second. It would be most valuable if we could learn in what manner the rate of mortality influences the values of a policy, and how to ascertain, if possible, whether a given Table will produce a larger or smaller value than any other Table, otherwise than by the actual process of working out the results from each.

When the values of the annuities by each Table are given it is easy to ascertain whether the value of a policy by one Table will be greater or less than by another. Thus, suppose we have two Tables of mortality given:

Let $V_{x|n}$ and $V'_{x|n}$ represent the values of a policy, taken out at the age x and kept in force for n years, by the first and second Tables; and

Let a_x, a'_x represent the values of the annuities at age x by the same Tables, respectively;

Then, since
$$V_{x|n}=1-\frac{1+a_{x+n}}{1+a_x},$$

and
$$V'_{x|n}=1-\frac{1+a'_{x+n}}{1+a'_x},$$

therefore
$$V_{x|n}>=<V'_{x|n}$$

according as
$$1-\frac{1+a_{x+n}}{1+a_x}>=<1-\frac{1+a'_{x+n}}{1+a'_x},$$

as
$$\frac{1+a'_{x+n}}{1+a'_x}>=<\frac{1+a_{x+n}}{1+a_x},$$

that is, the values of a policy by one Table will be greater or less than by another Table according as the values of the annuities-due decrease in a greater or less ratio.

If $\frac{1+a'_{x+n}}{1+a'_x}$ be greater than $\frac{1+a_{x+n}}{1+a_x}$, then $\frac{a'_{x+n}}{a'_x}$ will generally be greater than $\frac{a_{x+n}}{a_x}$, and we might safely say that a Table, whose annuity values decrease in a greater ratio than those by another Table, will give the greater policy values. From this it might reasonably be inferred that a Table in which the mortality *increases at a greater rate* than represented by another Table would give the larger value for a policy, even though the latter Table might exhibit a greater mortality. In order to see whether this really would be the case, I formed two hypothetical Tables of Mortality by taking the Carlisle premiums, and in one case adding a constant quantity throughout, and in the other case increasing them by 25 per cent; and then working out the rate of mortality by an inverse process. I thus obtained two Tables, each with a higher mortality throughout than the "Carlisle"; while at the same time one Table would give a smaller, and the other a larger Policy value throughout. The result was very unfavourable to the theory; for while the rate of mortality throughout the first hypothetical table increased at a slower rate than in the Carlisle table; in the second, instead of increasing at a greater rate, the ratio was generally below that in the original Table, and, towards the end, even fell below the other hypothetical Table. So that, although it may generally be the case that a Table, in which the mortality increases at a greater rate than in another Table, will give a greater Policy Value, yet it will not always be so; and, therefore, can be no criterion to go upon. Nor are there any means, so far as I can at present see, of ascertaining, with certainty, what influence the rate of mortality has upon the value of a Policy.

Conditions of the Messenger Prize (1868) (*above referred to*).

INSTITUTE OF ACTUARIES OF GREAT BRITAIN AND IRELAND.

The Council of the Institute of Actuaries have resolved again to offer a Prize of the value of Ten Guineas for the best Essay to

be written by a Member of the Institute who has passed his Second (or Third) Year's Examination, and is not an advertised officer of any Insurance Company. The Prize will be given in the form of books, to be selected by the successful competitor from works relating to Mathematical, Statistical, or Economical Science.

The subject proposed on the present occasion is as follows:

"A Comparison of the Values of Policies as found by means "of the various Tables of Mortality and the different "methods of Valuation in use among Actuaries."

In the treatment of this subject, it is desired that the competitors should—

(1) Enumerate the various tables of mortality, and give references to the works containing the original data on which each is founded.

(2) Describe the different methods of valuation now employed, so far as known.

(3) Give examples in a tabular form of the values of policies taken out at various ages, and continued in force for various terms, as found from each table of mortality, each rate of interest, and by each method of valuation.

(4) Consider to what extent the employment of any particular data or method may affect the reserve made by a Life Office for its liabilities. Under this head it is desired that examples should be given corresponding as nearly as may be to the conditions which are found to prevail in practice.

(5) Give illustrations of the manner in which the divisible surplus, present and future, will be influenced by the same circumstances.

(6) Examine in what manner the rate of mortality influences the value of the policy; showing in what cases a mortality table with a high rate of mortality gives a larger value, and in what cases a smaller value, than another table in which the rate of mortality is lower.

CONDITIONS OF THE COMPETITION.

1. That the Essays shall be sent in to the President of the Institute on or before the 1st of January next.

2. That the names of the competitors shall be sent in under seal, with a Motto corresponding to one to be affixed to the head of the Essay, such Motto and Essay not to be in the handwriting of the competitor.

3. The Essay for which a prize shall be awarded to become the property of the Institute.

4. The unsuccessful Essays to be returned with the corresponding envelopes unopened.

5. No prize shall be awarded unless the adjudicators shall consider some Essay worthy of the distinction.

By Order of the Council,

ARTHUR H. BAILEY,
ARCHIBALD DAY,
Honorary Secretaries.

12, St. James's Square, S.W.
1st June, 1867.

On the Construction of Tables by the Method of Differences. By Peter Gray, F.R.A.S., *Honorary Member of the Institute of Actuaries.*

Section IV.—*On the Construction of Tables in which the Characteristic Function is Irrational.*

(118). Irrational Functions, strictly speaking, are functions whose numerical values cannot be expressed by finite fractions. In the present connexion the term irrational has a wider signification. It is used to denote any function whose numerical value cannot, in general, be expressed *exactly* within the limits as to decimal places, to which we restrict ourselves in the table under formation. As here employed therefore the term designates not only transcendental functions, (as exponential, logarithmic, circular, &c.,) and algebraical irrational functions, (which are such as contain fractional powers of the variable in either numerator or denominator, or both,) but also algebraical fractional functions, which are such as contain integer powers of the variable in both numerator and denominator, or in the latter only. And it is to the formation of tables of the values of functions such as these that our attention is now to be directed.

(119). The characteristic of irrational functions which renders necessary a different mode of treatment from that which we found so efficient when the functions dealt with were rational, is that those functions have no constant differences. It was shown (60) that in a rational function of the nth degree the nth difference is

constant, so that those of the $(n+1)$th and all higher orders vanish. The consequence of this with which we are here concerned is that where a single value of a function of the nth degree, and its differences up to the nth inclusive are given, we can readily form by means of them, and continue to any extent desired, a series of successive values of the function. Now irrational functions when expanded take the form of infinite series: their degree is therefore infinite, and consequently their differences never vanish. It is obvious then that we cannot in regard to them adopt the same course as in the case of rational functions. It is conceivable however, and it will presently be shown to be the fact, that, neglecting all beyond, we may assume a difference, of an order depending on the specialties of the case, to be constant for a definite interval, without producing sensible error in the values formed upon this hypothesis. The operation of construction will thus become assimilated in a measure to that with which we are already familiar, the chief point of distinction being that here we shall have to form a new series of differences corresponding to every change that we make in the value of the final difference, which is treated, for a certain interval, as constant.

(120). In forming a table of the values of an irrational function our basis of operations, so to speak, is a series of equidistant values of the function, the intervals being greater than those that are to be between the values constituting the table. Those values, which we call the fundamental or primitive values, may either be given, or may have to be formed by aid of the characteristic function, when it is known; and from the differences derived from them we form, by application of the formulæ in (78), the differences pertaining to the series to be constructed. After this the operation proceeds pretty much as in the case of rational functions, with, however, certain modifications, which will be exhibited and explained in due course.

(121). It is a necessary preliminary in our operations that the differences of the series of primitive values should converge, that is, that as they ascend in order they should decrease in magnitude; and it is obvious that the more rapidly they converge, the fewer orders of differences will it be necessary to take account of, as we shall thus the sooner reach an order small enough to be neglected. When the characteristic function is given we have it in our power usually to obtain the requisite degree of convergence in the differences, by forming primitive values sufficiently close. That the differences are more convergent for small intervals than for large

will appear by inspection of the formulæ in (78). Thus, if of two series (commencing at the same point) of equidistant values of the same function, the interval between successive terms of the first be m times that between successive terms of the second, then, if the differences of the first be

$$\Delta^1, \Delta^2, \Delta^3 \ldots \Delta^r,$$

those of the second will be, *approximately*,

$$\frac{\Delta^1}{m}, \frac{\Delta^2}{m^2}, \frac{\Delta^3}{m^3} \ldots \frac{\Delta^r}{m^r}$$

respectively. And these, m being greater than unity, are obviously more convergent than the others.

(122). It will have been already perceived, and it will become more apparent hereafter, that the labour attending the use of a considerable number of orders of differences in construction is not small; and it increases in a rapidly increasing ratio, as that number is increased. There are two methods of avoiding, as far as may be, the labour attending the use of many orders of differences. One of them was hinted at in (121); and it consists in forming fundamental values so close that the *final* difference it is necessary to take account of shall not be of a high order. This method is available up to a certain point; that is till we have got the orders of differences it is necessary to take account of compressed into a manageable number—say four, five or six, according to the extent of the table to be formed. To go beyond this would be to incur a portion of the very labour which the use of the method of differences is designed to supersede. And therefore, when this point has been reached we have recourse to the other method to which reference has been made, and which I now proceed to describe.

(123). Suppose that a table has to be constructed comprising 100,000 values of a given function; and suppose also that, having formed each thousandth value, we find on differencing the one hundred values so formed, that the fifth order of differences is so nearly constant that those of the sixth and higher orders may be neglected. It might appear that we could now proceed to the completion of the table by the insertion of 999 terms in each interval. We certainly could; but there are objections all but insuperable to such a course. The construction of 100,000 terms by the use of five orders of differences would, in the most favourable circumstances, be a sufficiently formidable undertaking, while here the circumstances are about as unfavourable as they could well be. The cause of this is the comparative magnitude of the

intervals in the fundamental series and in the series to be formed. The ratio is 1000 : 1; hence to form the differences to be used in the construction, m, in the formulæ of (78), will have to be made 1000, and it is apparent that, as a consequence, they would extend seventeen places at least to the right of the last place in the fundamental values. It is not easy to form a just estimate of the immense mass of figures, useless in the final results, that we should thus be compelled to write. Add to this that we should have no verification of our work as it proceeds, but on the attainment of each thousandth term of the series being formed, and it is quite plain that nothing further need be said to show the desirableness of avoiding, if possible, this way of going to work.

(124). The remedy for these inconveniences is to effect the required subdivision of the intervals in the fundamental series by *successive* operations. Thus, in each interval, instead of 999, I should first insert 9 terms; in each interval of this new series I should again insert 9 terms, and so on till the table should be completed. By conducting the process in this way the number of *extra* figures is very much reduced, as is also the number of orders of differences necessary to be taken account of. In the first interpolation only, comprising 1000 terms, should we have to use so many as five orders; for the second, (10,000 terms,) four—perhaps three—would suffice; and for the third, (100,000) two orders would be sufficient; and this last, it will be found, can be made a very simple operation. In all these operations also, it will be perceived that we have a very effectual check on the formation of every tenth term.

(125). I have *said* that in certain circumstances *small* differences may be neglected without producing sensible effect on the resulting series. It is time now to *show* in what circumstances this may be done; and also *how small* the differences must be in order that it may be done with safety.

(126). The expression for the nth term of a series whose initial term is u_x is (70),

$$u_{x+n}=u_x+n\Delta u_x+\frac{n(n-1)}{1.2}\Delta^2 u_x+\frac{n(n-1)(n-2)}{1.2.3}\Delta^3 u_x+\ldots,$$

the interval between successive terms being unity, to be afterwards subdivided by giving to n fractional values (76).

The contribution of a specific difference, say the rth, to u_{x+n} is,

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}\Delta^r u_x,$$

which for convenience we may write

$$\frac{\phi(n)}{1.2\ldots r}\Delta^r u_x.$$

Now the greatest error committed by neglecting this difference will be in that term or in those terms corresponding to that value or those values of n which give to $\phi(n)$ the greatest *numerical* value, whether positive or negative. Such value or values of n will evidently have to be sought amongst those which render $\phi(n)$ a maximum or a minimum;* and we shall therefore have to examine $\phi(n)$ with a view to determine its maxima and minima, for the integer values of r from 2 to 7, the sixth being the highest order of differences we have made provision for using.

(127). To determine the values of the variable which render an algebraical function of the rth degree a maximum or a minimum we require generally to solve an equation of the $(r-1)$th degree. This would be rather a tedious and troublesome operation in the case of values of r so great as some of those we have to deal with. It is fortunate therefore that here, owing to the peculiar form of $\phi(n)$, we are able to effect our object by the solution of an equation in each case of not more than half the dimensions of that which, but for this peculiarity of form, would have been necessary.

(128). The property of which advantage is thus taken may be enunciated as follows:—

If $\phi(n)$ be a function of n of the form

$$n(n-1)(n-2)\ldots\ldots(n-r+1),$$

the substitution in it for n of $\frac{1}{2}(r-1)+a$, (a being any positive number, rational or irrational,) will give the same result *numerically* as the substitution of $\frac{1}{2}(r-1)-a$; and the signs of the results will be the same or different according as n is an even or an odd number.†

* It may be necessary to explain that a maximum is not necessarily the greater nor a minimum the least value that a function can assume. A function of the nth degree has usually $n-1$ maxima and minima values; and their characteristics are that a maximum is greater than preceding and following values, and a minimum is less. Thus, if a be a value of x that renders $\phi(x)$ a maximum, we must have $\phi(a)$ *greater* than both $\phi(a+h)$ and $\phi(a-h)$; and if a renders $\phi(x)$ a minimum we must have $\phi(a)$ *less* than both $\phi(a+h)$ and $\phi(a-h)$, h being in both cases supposed indefinitely small. It is obvious then, that it is *amongst* the maxima and minima values of a function that the greatest numerical value it can assume must be sought.

† The above is a case of the following more general theorem:—If $\phi(x)$ be a function of the nth degree whose roots are

$$0, t, 2t, 3t\ldots\ldots(n-1)t,$$

and if $k=\frac{1}{2}(n-1)t$, substitution in $\phi(x)$ of $k+a$ and $k-a$ successively, for x (a being any number, rational or irrational), will give results equal in numerical value, and of the same or different signs, according as n is an even or an odd number. That is, we shall always have, in a function constituted as above,

$$\phi(k+a)=(-1)^n\phi(k-a).$$

This theorem, which I do not recollect to have met with anywhere, is very easily proved.

(129). I do not here offer any demonstration of this property, but content myself with giving one or two numerical illustrations. Thus, if r (which is the number of factors, and consequently denotes the degree of $\phi(n)$), be 5, then $\frac{1}{2}(r-1)=2$; and if we take 6 for a, we have

$$\begin{aligned} \phi(2+6) \text{ or } \phi(8) &= 8\ .\ 7\ .\ 6\ .\ 5\ .\ 4 \\ \phi(2-6) \text{ or } \phi(-4) &= -4.-5.-6.-7.-8 \end{aligned}$$

Again, r still being 5, let $a=\frac{3}{4}$.

$$\begin{aligned} \therefore \phi(2+\tfrac{3}{4}) \text{ or } \phi(2\tfrac{3}{4}) &= 2\tfrac{3}{4}\,.\,1\tfrac{3}{4}\ .\ \tfrac{3}{4}.-\ \tfrac{1}{4}.-1\tfrac{1}{4} \\ \phi(2-\tfrac{3}{4}) \text{ or } \phi(1\tfrac{1}{4}) &= 1\tfrac{1}{4}\ .\ \tfrac{1}{4}.-\tfrac{3}{4}.-1\tfrac{3}{4}.-2\tfrac{3}{4}. \end{aligned}$$

It is needless to perform the multiplications here indicated. The two sets of factors in each example are the same numerically, and so also consequently will be the products. These moreover will differ in sign, r being in both examples an odd number. If in like manner an example be formed in which r is an even number, the numerical identity of the factors for any value of a, and consequently of the products, will still subsist; and these last will be in this case of the same sign.

(130). The manner in which the property just explained is made available for the reduction of the degree of the equation to be solved is as follows:—If, writing k for $\frac{1}{2}(r-1)$, $k+a$, $k+\beta$, $k+\gamma$, &c. are values of n which correspond to maxima or minima values of $\phi(n)$, so also are $k-a$, $k-\beta$, $k-\gamma$, &c.; since, as has just been shown, the results of the substitution in $\phi(n)$ of the latter set of values are the same *numerically* as those of the substitution of the former. But, by the Theory of Equations, the values of n which render $\phi(n)$ a maximum or a minimum are the roots of the *derived function* of $\phi(n)$ (the manner of forming which will be presently shown); hence this function, which is denoted by $\phi'(n)$, will admit of being written

$$\phi'(n)=\mathrm{A}\{(n-\overline{k+a})(n-\overline{k-a})(n-\overline{k+\beta})(n-\overline{k-\beta})\ .\ .\ .\ .\ \},$$

in which A is the coefficient of the highest power of n. The last expression may also obviously be written thus,

$$\phi'(n)=\mathrm{A}\{(\overline{n-k}-a)(\overline{n-k}+a)(\overline{n-k}-\beta)(\overline{n-k}+\beta)\ .\ .\ .\ .\ \};$$

from which it appears that if we transform $\phi'(n)$ into $\phi'(n+k)$ we shall obtain

$$\phi'(n+k)=\mathrm{A}\{(n-a)(n+a)(n-\beta)(n+\beta)\ .\ .\ .\ .\ \},$$

an expression whose roots are obviously $+a$, $-a$, $+\beta$, $-\beta$, &c.

If moreover we multiply the factors of this expression in pairs it will become

$$\phi'(n+k)=A\{(n^2-\alpha^2)(n^2-\beta^2)\ldots\ldots\};$$

and hence it appears that, whatever be the degree of $\phi'(n)$, if we transform it into $\phi'(n+k)$, k being known and equal to $\frac{1}{2}(r-1)$, (where r is the number expressing the degree of $\phi(n)$,) it must and it will assume *the form* of a function of half the degree of $\phi'(n)$, and whose roots are α^2, β^2, &c. And these roots being found those of $\phi'(n)$ are known.

(131). It will be observed that the roots of $\phi'(n)$ go in pairs, $k\pm\alpha$, $k\pm\beta$, &c. This is intelligible when $r-1$, the degree of $\phi'(n)$, is an even number; but it is less so, at first sight, when $r-1$ is an odd number. The explanation is that in the latter case one of the quantities α, β, &c., say α, will vanish, and the pair, $k\pm\alpha$, will coalesce into a single root, namely k.

(132). We are now prepared to enter upon an examination of the terms of the expansion of u_{x+n} in detail. I commence with the term involving $\Delta^7 u_x$. This term is

$$\frac{\phi(n)}{1.2\ldots 7}\Delta^7 u_x, \text{ or } \frac{\phi(n)}{5040}\Delta^7 u_x,$$

in which

$$\phi(n)=n(n-1)(n-2)\ldots\ldots(n-6);$$

and we have first to find the values of n which render this function a maximum or a minimum. The required values are, as stated (130), the roots of the derived function of $\phi(n)$. The derived function is formed as follows:—Multiplying out the factors of $\phi(n)$ we have

$$\phi(n)=n^7-21n^6+175n^5-735n^4+1624n^3-1764n^2+720n.$$

Now multiply each term of this by the exponent of n in the same term, and diminish the exponent in the result by unity. The aggregate of the terms so formed is the derived function.* Thus n^7 gives $7n^6$, $-21n^6$ gives $-126n^5$, and so on. We hence have

$$\begin{aligned}\phi'(n)&=7n^6-126n^5+875n^4-2940n^3+4872n^2-3528n+720\\&=7(n^6-18n^5+125n^4-420n^3+696n^2-504n+102{\cdot}857143).\end{aligned}$$

Now r being here equal to 7, $k=\frac{1}{2}(r-1)=3$. Therefore transform $\phi'(n)$ into $\phi'(n+3)$, as follows:—

* The function which in the Theory of Equations receives the name of the *derived function* of $\phi(n)$, is that which in the Higher Analysis is known as the *differential coefficient* of $\phi(n)$.

	1	−18	125	−420	696	−504	102·857143(3
		−15	80	−180	156	−36	−5·142857
		−12	44	−48	12	0	
		−9	17	+3	21		
		−6	−1	0			
		−3	−10				
		0					
Result,	1	*	−10	*	+21	*	−5·142857

That is,

$$\phi'(n+3)=7(n^6-10n^4+21n^2-5\cdot142857).$$

(133). This function may obviously be considered as of the third degree in n^2, and its roots, α^2, β^2, &c. evolved accordingly (130). I give the evolution of one of them, the greatest, say α^2, leaving the others as exercises for the student.

1	−10	21	−5·142,857(7·1718856
	−3	0*	−5·142
	+4	2800	−2·231857
	110	2911	−60044
	111	302300	−28206
	112	310259	−2726
	1130	31826$\cancel{7}$	−178
	1137	31838	−19
	1144	3185$\cancel{0}$	0
	$\cancel{1}1\cancel{1}1$		

The three roots are,

$$\alpha^2=7\cdot171886 \qquad \therefore\ \alpha=\pm2\cdot67804$$
$$\beta^2=2\cdot546521 \qquad \beta=\pm1\cdot59578$$
$$\gamma^2=\ \cdot281594 \qquad \gamma=\pm\ \cdot53065$$

And the roots of $\phi'(n)$, $k\pm\alpha$, $k\pm\beta$, &c., are consequently, arranging in order of magnitude and omitting superfluous figures,

·322	3·531
1·404	4·596
2·469	5·678

(134). The above roots of $\phi'(n)$ are, as stated (130), the values of n which correspond to maxima and minima values of $\phi(n)$, and we have to determine which of these maxima and minima are the greatest numerically, positive or negative. For this purpose we substitute the roots of $\phi'(n)$ in succession in $\phi(n)$, or rather the first three of them, since we know that the second three will give the same results as the first, in reverse order, and with opposite signs, r being here an odd number (128).

* What we learn from the occurrence of cipher here is that 7, the transforming number, is a root of the quadratic whose coefficients are those of the first three terms of $\phi'(n+3)$, viz., $n^2-10x+21$. The other root is obviously $10-7=21\div7$, $=3$. These however are matters with which we are not particularly concerned here.

The results are as follows, in order:—

95·84	−12·36
−23·15	23·15
12·36	−95·84

So that the greatest positive value of $\phi(n)$ is, say, 96, corresponding to $n=·322$, and the greatest negative value is -96, which corresponds to $n=5·678$.

(135). The term involving $\Delta^7 u_x$ is (132),

$$\frac{\phi(n)}{5040}\Delta^7 u_x;$$

substituting in which the last named values of $\phi(n)$, we have for the greatest error arising from the neglect of $\Delta^7 u_x$,

$$e=\pm\frac{96}{5040}\Delta^7 u_x=\pm·01905\Delta^7 u_x.$$

(136). Perhaps the foregoing investigation and its results will be rendered more intelligible by a geometrical illustration.

Let the *continuous* curve in the adjoining figure be the *locus* of the equation

$$y=u_{x+n},$$

where u_{x+n} is of the form

$$u_x+n\Delta u_x+\frac{n(n-1)}{2}\Delta^2 u_x+\ldots.+\frac{n(n-1)(n-2)\ldots(n-6)}{1.2\ldots7}\Delta^7 u_x,$$

terms involving $\Delta^8 u_x$, &c., being supposed too small to have any sensible influence between the limits $n=0$ and $n=6$. That is, any value of n being measured from 0 (the *origin*) along the *axis*, an *ordinate* erected from its extremity, meeting the curve, is the corresponding value of y. Thus, the ordinates drawn from the

points 0, 1, 2, &c., will represent u_x, u_{x+1}, u_{x+2}, &c., the terms from which the differences, Δu_x, &c., are derived. Now the investigation shows that if the curve be drawn corresponding to the same equation, omitting the term involving $\Delta^7 u_x$, it will be *of the character* shown by the waved and dotted curve. Since the omitted term vanishes for integer values of n from 0 to 6, inclusive, the two curves will intersect at the points whose *abscissæ* have these values; but between those points the second curve will deviate from the first alternately on the one side and the other, since for points situated in adjoining intervals the coefficient of $\Delta^7 u_x$ is alternately a maximum and a minimum (133). I have indicated by the dotted ordinates the position of the points in the curve where the maxima and minima deviations occur. They are those corresponding to the values of n in (133), namely ·322, 1·404, &c., one in each interval. The portions of these ordinates intercepted between the two curves, are the measures of the deviations at the points in question; and it is shown in (134) that the two points of greatest deviation are those corresponding to $n = \cdot 322$, and $n = 5\cdot 678$,* the deviation at the former point being positive and that at the latter negative, supposing $\Delta^7 u_x$ positive. Finally I have determined in (135) the magnitude of the deviation at the points in question.

(137). We have now to investigate the effect of neglecting $\Delta^6 u_x$, differences of the seventh and higher orders being here supposed to be too small to be taken account of between the limits $n=0$, and $n=5$.

The term involving $\Delta^6 u_x$ is

$$\frac{\phi(n)}{1.2\ldots 6}\Delta^6 u_x, \text{ or } \frac{\phi(n)}{720}\Delta^6 u_x,$$

in which

$$\phi(n) = n(n-1)(n-2)\ldots(n-5)$$

$$= n^6 - 15n^5 + 85n^4 - 225n^3 + 274n^2 - 120n;$$

and the derived function is, (132),

$$\phi'(n) = 6n^5 - 75n^4 + 340n^3 - 675n^2 + 548n - 120.$$

* This would be apparent from the figure were it correctly drawn, and on a sufficiently large scale. But it is not correctly drawn, the dotted curve being sketched by hand. And the deviations *in the figure* are not necessarily greatest at the points indicated. Also, from the necessity of the case, the dotted curve is so drawn that the portions of it situated between the points of intersection with the continuous curve are alternately concave and convex to the axis of abscissæ, in consequence of which the points in question become *points of contrary flexure.* Neither is this necessarily the case. It will only be so when the neglected term in u_{x+n} is comparatively large. It is easy to conceive, though very difficult to draw, a curve which, while lying alternately on opposite sides of the given curve, shall yet, like it, be convex to the axis throughout.

Now here $r=6$. Hence $k=\frac{1}{2}(r-1)=2{\cdot}5$, and we transform $\phi'(n)$ into $\phi'(n+2{\cdot}5)$ thus:—

	6	−75	340	−675	548	−120(2·5
		−60	190	−200	48	0
		−45	77·5	−6·25	32·375	
		−30	2 5	0		
		−15	−35			
		0				
Result,	6	*	−35	*	32·375	*

That is,

$$\phi'(n+2{\cdot}5)=6n^5-35n^3+32{\cdot}375n,$$

a root of which is obviously 0, say $a^2=0$, (131); and the remaining roots are those of

$$6n^4-35n^2+32{\cdot}375,$$

which may be treated as a quadratic in n^2, as follows:—

6	−35	32·3750(1·1528323
	−29	3·37
	−230	1·1350
	−224	60000
	−2180	17624
	−2150	683
	−21200	48
	−21188	6
	−21176	

The other root may be either developed in the same way, or got by subtracting that just found from 35÷6, (the quotient of the second coefficient by the first, with its sign changed,) $=5{\cdot}8333333$.

(138). We thus have for the roots of $\phi'(n+2{\cdot}5)$,

$$\begin{array}{ll} a^2=0 & \therefore\ a=\pm 0 \\ \beta^2=1{\cdot}152832 & \beta=\pm 1{\cdot}07494 \\ \gamma^2=4{\cdot}680501 & \gamma=\pm 2{\cdot}16344; \end{array}$$

and the roots of $\phi'(n)$, being $k\pm a$, $k\pm\beta$, &c., are consequently,

·337	2·500	3·575
1·425		4·663

These are the values of n which correspond to maxima and minima values of $\phi(n)$; and it will be remarked that, like those determined in (133), they fall very near the middle of the intervals which separate the given values, u_x, u_{x+1}, &c. Their places, were the intervals divided into ten parts, would be in the neighbourhood of the 3rd, 14th, 25th, 36th, and 47th terms.

(139). Now substituting the above values successively in $\phi(n)$ we get,

$$\begin{array}{ccc} -16\cdot90 & -3\cdot52 & 5\cdot05 \\ 5\cdot05 & & -16\cdot90 \end{array}$$

Hence the term involving $\Delta^6 u_x$ being, (137),

$$\frac{\phi(n)}{720}\Delta^6 u_x,$$

the greatest value it can assume, and consequently the greatest error committed by neglecting this term will be,

$$e = -\frac{16\cdot90}{720}\Delta^6 u_x = -\cdot02347\Delta^6 u_x.$$

(140). The term involving $\Delta^5 u_x$ is,

$$\frac{\phi(n)}{1.2\ldots5}\Delta^5 u_x, \text{ or } \frac{\phi(n)}{120}\Delta^5 u_x,$$

where $\qquad \phi(n) = n(n-1)(n-2)\ldots.(n-4),$

$$= n^5 - 10n^4 + 35n^3 - 50n^2 + 24n.$$

The derived function is, (132),

$$\phi'(n) = 5n^4 - 40n^3 + 105n^2 - 100n + 24,$$

or $\qquad \phi'(n) = 5(n^4 - 8n^3 + 21n^2 - 20n + 4\cdot8).$

Here r being equal to 5, k or $\frac{1}{2}(r-1) = 2$. Therefore transform $\phi'(n)$ into $\phi'(n+2)$ as follows:—

$$\begin{array}{rrrrrl} & 1 & -8 & 21 & -20 & 4\cdot8(2 \\ & & -6 & 9 & -2 & \cdot8 \\ & & -4 & 1 & 0 & \\ & & -2 & -3 & & \\ & & 0 & & & \\ \hline \text{Result,} & 1 & * & -3 & * & \cdot8 \end{array}$$

Hence, $\qquad \phi'(n+2) = 5(n^4 - 3n^2 + \cdot8);$

and this may be treated as a quadratic in n^2, as follows:—

$$\begin{array}{lll} 1 & -30^* & 80(\cdot29584054 \\ & -28 & 2400 \\ & -260 & \;14100 \\ & -251 & \;\;202500 \\ & -2420 & \;\;\;\;9764 \\ & -2415 & \;\;\;\;\;130 \\ & -24100 & \;\;\;\;\;10 \\ & -24092 & \\ & -2408\cancel{4} & \end{array}$$

The other root will be got by a similar development, or by sub-

* The roots are multiplied by 10 at the outset, the first root figure here being in the first decimal place (32).

tracting that just found from 3, the second coefficient with its sign changed.

(141). We thus have,

$$\alpha^2 = \cdot 29584 \qquad \therefore \alpha = \pm \cdot 54391$$
$$\beta^2 = 2\cdot 70416 \qquad \beta = \pm 1\cdot 64444$$

And the roots of $\phi'(n)$, being $k \pm \alpha$, $k \pm \beta$, are consequently,

·356	2·544
1·456	3·644

(142). Now substituting these values successively in $\phi(n)$ we get,

3·63	1·42
−1·42	−3·63

Hence the term involving $\Delta^5 u_x$ being (140),

$$\frac{\phi(n)}{120} \Delta^5 u_x,$$

the greatest error consequent on the omission of this term will be,

$$e = \pm \frac{3\cdot 63}{120} \Delta^5 u_x = \pm \cdot 03025 \Delta^5 u_x.$$

(143). The term involving $\Delta^4 u_x$ is

$$\frac{\phi(n)}{1.2.3.4} \Delta^4 u_x, \text{ or } \frac{\phi(n)}{24} \Delta^4 u_x,$$

in which

$$\phi(n) = n(n-1)(n-2)(n-3)$$
$$= n^4 - 6n^3 + 11n^2 - 6n.$$

The derived function is (132),

$$\phi'(n) = 4n^3 - 18n^2 + 22n - 6$$
$$= 2(2n^3 - 9n^2 + 11n - 3).$$

Here $r=4$, and k or $\frac{1}{2}(r-1) = 1\cdot 5$. Transform $\phi'(n)$ into $\phi'(n+1\cdot 5)$ as follows:—

	2	−9	11	−3(1·5
		−6	2	0
		−3	−2·5	
		0		
Result,	2	*	−2·5	*

That is, $\phi'(n+1\cdot 5) = 2(2n^3 - 2\cdot 5n)$,

a root of which is $\alpha^2 = 0$, and the remaining roots are contained in $2n^2 - 2\cdot 5$, or $n^2 - 1\cdot 25$. This gives,

$$\beta^2 = 1\cdot 25 \quad \therefore \beta = \pm 1\cdot 1180.$$

(144). Hence the roots of $\phi'(n)$, $k\pm\alpha$, $k\pm\beta$, are

$$\begin{array}{r} \cdot 382 \\ 1\cdot 500 \\ 2\cdot 618 \end{array}$$

And substituting these in $\phi(n)$ we obtain,

$$\begin{array}{r} -1\cdot 00 \\ 0\cdot 45 \\ -1\cdot 00 \end{array}$$

Hence the greatest error committed by neglecting a fourth difference will be

$$e=-\frac{1}{24}\Delta^4 u_x=-\cdot 04167\Delta^4 u_x.$$

(145). The term involving $\Delta^3 u_x$ is,

$$\frac{\phi(n)}{1.2.3}\Delta^3 u_x, \text{ or } \frac{\phi(n)}{6}\Delta^3 u_x,$$

in which
$$\begin{aligned}\phi(n)&=n(n-1)(n-2)\\ &=n^3-3n^2+2n;\end{aligned}$$

and the derived function of this is, (132),

$$\phi'(n)=3n^2-6n+2.$$

Here $r=3$, and $k=\frac{1}{2}(r-1)=1$. Transform therefore $\phi'(n)$ into $\phi'(n+1)$ thus:—

$$\begin{array}{lccc} & 3 & -6 & 2(1 \\ & & -3 & -1 \\ & & 0 & \\ \hline \text{Result,} & 3 & * & -1 \end{array}$$

That is,
$$\phi'(n+1)=3n^2-1;$$

which gives,
$$\alpha^2=\frac{1}{3} \quad \therefore \alpha=\pm\cdot 57735.$$

(146). Hence the roots of $\phi'(n)$ are

$$\cdot 4226 \text{ and } 1\cdot 5774;$$

and substituting these in $\phi(n)$ we have

$$\cdot 385 \text{ and } -\cdot 385.$$

Hence the greatest error committed by neglecting $\Delta^3 u_x$ will be,

$$e=\pm\frac{\cdot 385}{6}\Delta^3 u_x=\pm\cdot 06417\Delta^3 u_x.$$

(147). The term involving $\Delta^2 u_x$ is,

$$\frac{\phi(n)}{2}\Delta^2 u_x,$$

where
$$\phi(n)=n(n-1)=n^2-n.$$

The derived function of this is, (132),

$$\phi'(n)=2n-1,$$

which needs no transformation, but gives at once the root of $\phi'(n)=\cdot 5$. And substituting this for n in $\phi(n)$ we get,

$$\phi(\cdot 5)=-\cdot 25.$$

Hence the term involving $\Delta^2 u_x$ becomes,

$$e=-\frac{\cdot 25}{2}\,\Delta^2 u_x=-\cdot 125\Delta^2 u_x.$$

(148). Collecting our results, we have found for e, the value of the greatest error committed by neglecting a difference of each order from the second to the seventh, as follows:—

$$e=\cdot 1250\Delta^2 u_x, \qquad e=.0303\Delta^5 u_x,$$
$$e=\cdot 0642\Delta^3 u_x, \qquad e=.0235\Delta^6 u_x,$$
$$e=\cdot 0417\Delta^4 u_x, \qquad e=\cdot 0191\Delta^7 u_x.$$

(149). But these results will be more convenient for use if put in the following form. Solving the several equations for $\Delta^2 u_x$, &c., we get

$$\Delta^2 u_x=\ 8e, \qquad \Delta^5 u_x=33e,$$
$$\Delta^3 u_x=16e, \qquad \Delta^6 u_x=43e,$$
$$\Delta^4 u_x=24e, \qquad \Delta^7 u_x=52e.$$

It is now apparent that, having determined on the greatest error that can be tolerated in the values to be formed, we can, by substituting that error in the expression for any difference that may be in question, at once learn whether that difference may be safely neglected or not. Thus if, for example, an error of unit in the last place of the values to be formed be admissible, and if the differences of the sixth and higher orders are inappreciable, then the fifth difference also may be neglected if it do not exceed 33. And similarly of the other differences in the table.

(150). There is another source of error in the interpolation of irrational functions. The fundamental values are not absolutely correct; they are correct only to the nearest unit in the last place. In consequence the interpolated values may, and occasionally will, err by as much as a unit in the last place. If moreover these values, with their possible errors, are used as fundamental values for further interpolation, it is obvious that any errors in them will be reproduced, in a somewhat aggravated form, in the new values. A very simple and efficient remedy exists for this. It is to carry our computations to one, two, or three places* (according to the

* These extra places are irrespective of, and in addition to, those arising in consequence of the fractional values of the several differences in (81).

degree of accuracy required) beyond the number that are to be retained and tabulated. The effect is, that in computing to two places beyond the number required for tabulation, our knowledge of the possibility of an error of say 3, in the last of them, will only leave us in doubt, in the few cases in which the extra figures are between 47 and 53, whether the usual conventional increase is due to the last figure retained or not. And if we go to three extra places the doubtful cases, for a possible error of 3, will be between the limits 497 and 503. Hence if the facility for the occurrence of the possible error for all points of the series were the same, we should probably have in the first case, three instances in every hundred of the values formed, and in the second three in every thousand, in which the increase is wrongly given or withheld. Such an amount of liability to error as this will not disturb the equanimity of either the most timid computer, or the greatest stickler for accuracy.

(151). The preliminary matter being now disposed of, we are ready to enter upon examples of the actual construction of tables. The first example will be a table of Anti-logarithms, that is, of the numbers corresponding to all logarithms from 0 to 1.

The Mutual Life Insurance Company of New York.

ON the 1st February, 1868, this Company, "now the largest moneyed Corporation" in America, celebrated its "silver wedding," as it was termed by the Governor of the State of New York on the occasion of the 25th annual meeting. The Report then presented and the general Prospectus of the Company have been forwarded to us by the "President" of the Company, who appears to be virtually what in this country we should call "Managing Director." These documents are so instructive and interesting in themselves, and illustrate so remarkably the application to our pursuits of that originality of the American character, which is fast becoming as proverbial as its energy, that we propose to notice their contents in some detail.

Though most of us have a vague notion that the operations of American Insurance Companies are conducted on a scale of great magnitude, few will learn without astonishment that this particular company alone issued, in its last financial year, no fewer than 19,406 Policies, for sums amounting in the aggregate to over

twelve millions sterling ($62,252,606), on which more than *eight hundred thousand pounds* ($4,007,064 10) were paid as "original or first year's premiums." There is danger of our failing to grasp the meaning of these figures. They mean that a new business was brought together, in one year, double and even treble in amount the whole accumulated business of many of our oldest Offices, and, with but one or two notable exceptions, exceeding that of any Office in the United Kingdom, of whatever age or standing. It must, however, be borne in mind that a deduction is to be made from these and the other figures we shall give, on account of the depreciation of the paper currency of the United States.

At the date of the Report, the total business consisted of 52,384 Policies, assuring, with bonus additions, £38,868,323. To meet this liability there were assets to the amount of £5,063,864, and an Annual Income of £2,034,609, of which £1,771,457 was derived from premiums and £263,152 from interest on investments. The assets were made up of £4,532,490 actual funds and £531,374 miscellaneous items, chiefly unpaid premiums and interest, but including two that deserve notice, one being a sum of £99,988—equal to 10 per cent on the original investment—for "Market Value of Stocks in excess of cost," and one a sum of £109,451 for "Value of Future Commissions commuted"—an item of somewhat questionable propriety.

These Assets, it will be seen, are somewhat less than three years' premium Income, and less than 13 per cent of the gross Liabilities. This indicates a recent business; and indeed, the total Income has quintupled itself in the last four years. Since the close of the Civil War this Company, like most other American Institutions of the kind, has entered on a new and remarkable epoch. Year by year, during the last four years, the new premium Income has increased in almost geometric progression. In 1864, it was £55,805; in 1865, £109,116; in 1866, £230,813; in 1867, £435,388; in 1868, £801,413.

These almost incredible results have been achieved by an Office founded but 25 years ago, in a "rear room" in Wall Street, whose "new system," as it then was in America, "of Mutual Assurance" was regarded with so much apathy and distrust, "that a large number of the original Subscribers failed to take their Policies, and their places were supplied by others."

Results so extraordinary cannot have been brought about by ordinary means. By the adoption of 4 per cent as the rate of interest, the Company is certainly able to put before the American

public more tempting rates of premium for the earlier assuring ages than those to which the English public is accustomed. But this alone will not suffice for an explanation. There are, in our opinion, two principal causes—one due to the economic condition, and the other to the genius of the American people; the former being the rate of interest at which money can be improved, the latter, the introduction of what are known as "ten year non-forfeiture Policies," the peculiarity of which is that "if, after two annual premiums have been paid, further payments are to be discontinued, the holder may, upon due surrender of the original policy, in accordance with the rules of the Company, receive in lieu thereof a paid-up policy for as many tenth parts of the original sum insured as full annual premiums have been paid." To these causes the Prospectus before us adds a third, viz., the introduction of the system known here as that of "Endowment Assurance"—a system, the prospectus states, "better adapted to the wants of the American people than any other," as is shown, it says, by the fact that "of over £7,000,000 received last year in cash by the total number of American Companies . . . . . . we estimate over one half to be paid for Endowment premiums."

The New York Mutual Company adopted this system no longer ago than 1854, and that of non-forfeiture Policies so lately as 1858, and yet out of its total number of 52,384 Policies, 19,711 are on one or other of these plans or on a combination of the two. The growing and also the relative popularity of these systems may be seen in the details given of the business of 1867. Of its 15,993 new Policies, 7,431 only, or less than one half, were ordinary, continuous-premium assurances; of the remainder, 2,652 were life Policies on the non-forfeiture plan, 4,305 were ordinary endowments, and 1,605 ten-year endowments. We here see the cause of the high rate of premium—very nearly 6½ per cent—which is so noticeable in the returns of American Companies.

It cannot be doubted that neither of these systems carries out in its integrity the fundamental ideas of life assurance—(1) that assurance is for the benefit, not of the Life assured, but of those dependent on him; (2) that the best form of assurance is that which gives the largest contingent guarantee for the smallest immediate outlay. On the other hand there need be no difficulty in understanding why these two systems should be popular. The argument put forward for the ten-year system is in one view irresistible. There can be no doubt that there is "an obvious advantage in paying for an insurance during the productive period of life." Happily however this productive period is not limited to 5,

10 or even 15 years—the longest period, apparently, over which the payments under the non-forfeiture plan are made to extend. It is therefore possible, by making the productive period and the period of payment conterminous, to combine, in the highest degree, the two advantages of small premiums and the ability to pay them. It is to Assurance in this form that English Companies will have next to address themselves. As regards the Endowment system, the source of its popularity is obvious when we are told that "all Endowments that have thus far matured during life of the insured in this Company have proved equal to six per cent compound-interest investments, costing nothing for insurance or expense meanwhile."

Here we come upon what after all mainly accounts for the results we have detailed, upon that which has, in fact, been the main cause of the Company's success, viz., the rate of interest which it has been able to realize. In the paper by Mr. Sheppard Homans, to be found in the 11th vol. of this *Journal*, in which he explained the system devised by him for an equitable distribution of surplus, he stated that up to that time (1865) the net rate of interest realized by the New York Mutual Insurance Company had been 6½ per cent. It is now said to have been 7 per cent net over the whole 25 years. The rate current in recent years must therefore have been still higher. Let us see what the Company has consequently been able to do for its Policyholders. We cannot do better than transfer to our pages the particular illustration which the Company itself gives.

POLICY No. 487.

Amount $5000. *Dated February* 12*th*, 1844. *Age* 40.

ORDINARY LIFE PLAN. ANNUAL PREMIUM $160.00.

Dividend Dates.	No. of Premiums paid.	Amount of Premiums paid.	Cash Value of Dividends when made.	Percentage on Premiums.	Additions for Dividends.
Dividend of 1848,	4	$640 00	$174 30	27	$431 64
" " 1853,	5	800 00	227 66	28	499 59
" " 1858,	5	800 00	351 12	44	682 71
" " 1863,	5	800 00	783 30	98	1356 37
" " 1866,	3	480 00	589 64	123	956 27
" " 1867,	1	160 00	190 47	118	302 49
" " 1868,	1	160 00	201 04	126	312 81
	24	$3840 00	$2517 53	Average 65½	$4541 88

THE PAST.

Total Premiums paid in 24 years, ... $3840 00
Total Cash value of Dividends, ... 2517 53
Dividends average 65 per cent.

THE PRESENT.

Original Policy, ... $5000 00
Additions, February 12th, 1868, ... 4541 88

Total Insurance, ... $9541 88

THE FUTURE.

No more Premiums are required in cash. The Annual Dividends are likely to exceed the Annual Premiums. The excess will be applied at option of the assured, to increase Policy, or as an *annual cash income*.

This example needs no comment. It may be well, however, to add, by way of increasing our astonishment, if that be possible, that to meet their liability under this Policy they reserve $5094, $2175 on account of the sum assured and $2919 an account of its bonus additions, although the total amount received as premiums is only $3840! Speaking in general terms, the Company thus details the benefits they have been able to procure for their assured. "The experience of the Company thus far shows the extraordinary result, that after ten or twelve full annual premium payments on an ordinary Life Policy, the owner may keep the same in force without further cash payments, by simply *surrendering and slowly consuming* his former additions, while fifteen or eighteen full payments generally have sufficed to keep both policy and additions in force for the full amount by the annual cash dividends."*

It is not unnatural that they should themselves view their success with fear and trembling. They wisely warn their constituents "that if the rate of interest should fall in future below what it has been in the past less favorable results must be expected."

The Report contains what, to English eyes, appears a very novel feature. There is appended a mathematical demonstration of their method of dividing surplus. This plan is already familiar to actuaries, being a modified and simpler form of that given by Mr. Homans in the paper above referred to. It will be remembered that in that paper, the demonstration was for a quinquennial distribution, and sufficiently complicated it was. It may not, perhaps, be too much to say that the necessities of his plan compelled his Office to adopt their present system of annual divisions. By this means, at all events, much complication has been got rid of; and we freely confess at the same time that the method of annual divisions is far more suitable to Companies showing such rapid progress as the one we are considering, than it would be to the more steady-going English Offices. The resulting formula, by means of which the share of surplus attaching to each policy is determined, is this:

$$V_{x|n}(1+i')+(P_x-e)(1+i')-\frac{d_{x+n}}{l_{x+n}}(1-V_{x|n+1})-V_{x|n+1}=\chi_{x|n}$$

$$=\text{contribution to surplus.}$$

* The practice of the Company as to the surrender of the reversionary bonuses is, we believe, far more liberal than that of any English Office; for at any time, a sufficient portion of the bonus may be surrendered to pay the current premium.

Here $V_{x|n}$=the reserve fund at the beginning of the year of observation; P_x, the actual premium; e, the share of expenses chargeable on the Policy; i', the actual, as distinguished from i the assumed, rate of interest; and $\frac{d'_{x+n}}{l'_{x+n}}$, the contribution to the death fund of the year as deduced from the actual deaths amongst all the living of the particular age of the individual Policyholder. The surplus being mainly derived 1° from excess of actual over estimated interest, 2° from diminished mortality, and 3° from unexpended loading, the object of the method, as is well known, is to return to each Policyholder such portion of each of these items of surplus as rightly belongs to him. Stated in words, the plan is thus described in the *Journal* (vol. xi., page 124): "*Credit* each Policyholder, 1st with the amount actually reserved at the last preceding distribution of surplus as the then present value or re-insurance of the Policy; and 2nd, with the *effective* (or *full*) premiums paid since that time, both sums being accumulated at the actual current rate of interest to the date of the present distribution; and *charge* him, 1st, with the actual cost of the risk to which the Company has been exposed *during the interval*, determined by means of a table representing the rates of mortality and interest actually experienced; and, 2nd, with the amount now reserved as the present value of the Policy. The difference between the sum of his *credits* and the sum of his *debits* determines the *over-payment* or *contribution* from the Policy proper." Though not expressly stated here, it will be seen that the formula, in determining the cost of insurance, takes account only of the amount actually at risk, of the difference, that is, between the sum in the Policy and the reserve. By this means, to use their own words, "the cost of insurance may, and in many cases actually does, decrease each year, notwithstanding the increased age of the insured!"

The Author of this plan naturally claims for it a high degree of merit, and points, with excusable pride, to the fact that already twenty Companies in America follow it, adding that other Companies "will be compelled to adopt it as soon as the needed preparatory changes in their system can be made." We cannot, however, admit that he is at all justified in saying that "this method is now acknowledged by all the leading mathematicians of both parts of the world to be the only equitable plan of distributing surplus." On the contrary, judging from our recollections of the discussion that followed the reading of Mr. Homans's paper before the Institute of Actuaries, we should say that the method was

received with very little favour by English actuaries; and in evidence of this we have only to quote Mr. Jellicoe's words in summing up that debate:—"I think I may pronounce, *ex cathedrâ*, and at all events with a considerable degree of certainty, that you would accept the truth of the proposition that . . . . . . to give each person a share of the surplus in proportion to the number and amount of premiums he has paid in the interval, is really sufficient . . . . . . to deal out strict justice to all." And so far from there being any general consent that all other methods of distribution are inequitable, as would of course follow from the words we have quoted, it is well known that there are many actuaries who, looking at the matter in a broad sense, emphatically protest against branding as inequitable or untrue any method of distribution that is intelligible in itself and clearly put before those who are to be bound by it. It is after all (they say) a matter of contract. What may be mathematically true may be commercially undesirable. In the language of Professor De Morgan, quoted by Mr. Gray at a discussion at the Institute in March, 1864, "Whatever method is to be used, let it be known, and all will be right." Without going so far as this, many others are of opinion that widely different methods of distribution may be considered in every sense as equally equitable: as for example, that which appropriates all the profits to those policies on which the premiums received accumulated at compound interest exceed the sum assured, and that which makes the same proportionate addition to all policies becoming claims in the same official year.

For ourselves, while admitting that Mr. Homans's plan is a great improvement on the method which divides the surplus in proportion to the premium paid,—which method we learn from the prospectus prevails in most of the mutual Companies of the United States, and prevailed "in all till the introduction of the contribution plan of dividends by the Mutual Life in 1862"—we are of opinion that the method falls very far short of perfect equity, inasmuch as it makes the same charge for the current assurance of all the assured of the same age, whatever the length of time they have been on the Office books; and the assured of long standing thus share unduly, and therefore inequitably, in the profit from the reduced mortality consequent on the large influx of recently selected lives. Nor can the method be considered as satisfactory in other respects until a practical rule is given for the assessment of the proper share of the expenses to each policyholder. It is said "The equitable apportionment of the *expenses* of business

among the various policy-holders, especially when the sum total of the *contributions* [or over-payments] is greater than the total Surplus to be divided, is a difficult and delicate problem, upon which no specific directions can be given—the peculiarities of each Company requiring special treatment."

There are numerous peculiarities of practice in the New York Mutual Insurance Company—as compared with English Offices—which are well worth notice, but to which we can only briefly refer.

Their business is confined to what the Prospectus terms "the salubrious districts of the United States." It is stated emphatically that "risks in the South are not sought and no Agencies opened where extra premiums are necessary." The proposals are accepted, by the President alone, if, in his opinion, the evidence is "clear from all objections," or in consultation with the medical officers, if there be doubt. The maximum amount to be assured on one life is fixed at the small sum of £4,000 ($20,000); and, carrying boldly into practice a belief we in this country are as yet but learning to accept in theory, they charge, on ordinary life policies, for ages under 25, the same premium as for age 25. They moreover require an (apparently annual) extra premium of 10*s.* per cent on the sum assured for assurances on female lives between the ages 18 and 48, the childbearing period. The "Office age" is always taken as the age at the nearest (not the next) birthday. All policies issued, of whatever description, share in the profits—there are however apparently no short term policies granted.

On the other hand, we learn with surprise that the New York Mutual Office is singular among the American Offices in declaring reversionary bonuses on its policies.

The greatest care is shown in dealing with the Investments. "The Charter of the Company permits no speculation of any kind." United States Stock, Stock of the State of New York, and Bonds and Mortgages of real property in New York State, "worth in every case double the amount loaned," form their only securities. The Stock, they obtain by subscription and directly from the Government; all applications for loans on real estate, and all certificates of appraisers and others as to the value of the property to be pledged, are made *on oath;* whilst "to guard against any possible loss from the depreciation of property, the bonds and mortgages are drawn for one year, so that the Company can speedily collect its loans if necessary." Like caution, indicating

that there still lingers in the State that distrust of men in one another so peculiar to unsettled countries, is shown in the double audit to which the receipts and disbursements are subjected.

They seem to have anticipated Mr. Bunyon in what they term "the Instalment Feature," a system under which a claim by death is paid by fixed Instalments spread over a term of years to be agreed upon, the balance remaining with the Company at the end of each year receiving a stipulated rate of interest. The Company thus becomes "the Guardian or Trustee of the Survivors," and hence "the provision may be considered, humanly speaking, beyond any adverse contingency." These policies may be made "inalienable," if desired; as also may Policies granted to married women on the lives of their husbands under the authority of a special enactment of the State of New York, and by virtue of which such Policies are free from claims by the creditors or representatives of the Husband.

We are not surprised to find that in all points of settled usage this Company is behind us in liberality. "All premiums are due and payable at the office of the Company, in the City of New York," and though payment to an Agent is allowed, the privilege is fenced round with various restrictions that again mark distrust, and this provision appears to have operated very harshly during the late Civil War. No days of grace are allowed, and restoration of Policies is permitted "solely as an act of grace or courtesy, and when the interests of the Company will not be impaired in any way thereby." Applications for restoration "must invariably be accompanied by a certificate as to the health of the person whose life was insured, and at his expense, from a physician acceptable to the Company," whilst the Policy, if revived, becomes subject, "in accordance with the decision of the Internal Revenue Department, to stamp tax for the same amount as that required for a new Policy." This cost, also, the assured has to bear. Claims are paid sixty days after proof of death.

One reflection is forced on us by what has preceded. How long will it be before the Insurance tide which has so long set westward shall be rolled back on our own shores? When it comes, if it should come, how shall we be able to withstand it? We can do nothing for our Policyholders that will bear a moment's comparison with the results which this Company has accomplished and, we cannot doubt, will continue for many years to come to accomplish for them. At present we are safe in the merited distrust which hangs about the public credit of the country. But

old stains, if not renewed, will assuredly be worn out with time. Let its public credit be once established beyond reach of doubt or cavil, and America must become the savings bank of Europe, and certainly not least so in respect of such savings as take the form of Assurance premiums.

INSTITUTE OF ACTUARIES.

PROCEEDINGS OF THE INSTITUTE.—Session 1867–68.

First Ordinary Meeting, Monday, 25th November, 1867

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.
The following gentleman was elected a Fellow, viz.:—

Andrew Baden.

The President read a report of the Sixth International Statistical Congress held at Florence.

Mr. T. B. Sprague, M.A., read a paper by Mr. Jardine Henry, entitled "Memoir on Instrument for furnishing the D numbers, to four figures each, in Two-Joint-Life Annuity Tables on any basis."

Thanks having been voted to the President, Mr. Sprague, and Mr. Jardine Henry, the meeting adjourned to Monday, 30th December, 1867.

Second Ordinary Meeting, Monday, 30th December, 1867.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.
The following gentlemen were elected, viz.:—

Fellows.

Frederick J. Elderton.
William Wallis.

Associates.

Felix Bassett.
Thomas John Barnes.
Henry Milner Blundell.
David Carment.
David Deüchar.
Walter Maples Edwards.
Samuel Hunter.
Nathaniel Jellicoe.
Joseph Henry Mayor.
William Rae Macdonald.
Ronald McPherson.
Charles W. B. Oak.
Franklyn Pennington.
James Watson Rodger.
Clarence Smith.

Mr. J. Hill Williams read a paper "On Briggs's Method of Interpolation."

Thanks having been voted to Mr. J. Hill Williams, the meeting adjourned to Monday, 27th January, 1868.

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Third Ordinary Meeting, Monday, 27th January, 1868.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.
The following gentlemen were elected Associates, viz.:—

Richard Cooper Rundell.	Thomas Marr.
Edward Mantle.	F. G. Howell.

John M. Gandy.

The following was announced to be the result of the Examinations for 1867, viz.:—

MATRICULATION EXAMINATION.

Fourteen gentlemen sent in their names for this Examination, of whom two withdrew without giving in their papers, and three passed in the following order of merit, viz.:—

1. David Carment.
2. William Rae Macdonald.
3. Felix Bassett.

SECOND YEAR'S EXAMINATION.

Nine gentlemen sent in their names for this Examination, of whom one withdrew without giving in his papers and four passed, viz.:—

William Vaughan.	Thomas John Searle.
William Sutton, B.A.	Edward Smyth.

THIRD YEAR'S EXAMINATION.

Four gentlemen sent in their names for this Examination, of whom two passed, viz.:—

Henry Mountcastle.	Thomas Young Strachan.

Mr. Bailey read a paper by Mr. James Chisholm "On the arrangement of Commutation, or D and N, Tables."

Thanks having been voted to Mr. Bailey and Mr. Chisholm, the meeting adjourned to Monday, 24th February, 1868.

Fourth Ordinary Meeting, Monday, 24th February, 1868.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.
The following gentlemen were elected Associates, viz.:—

William Pierce Reynolds.	Henry Hansley.

The President announced that the "Messenger Prize" offered by the Council in June last for the best Essay "On a Comparison of the Values of Policies as found by means of the various Tables of Mortality and the different methods of Valuation in use among Actuaries," had been awarded to Mr. Henry William Manly, of the London and Provincial Law Assurance Society, and that it would be read at the March Meeting.

Mr. Makeham read a paper "On the Values of Annuities Certain."

Mr. Peter Gray read a paper "On the Rate of Interest in Loans repayable by instalments."

Thanks having been voted to Mr. Makeham and Mr. Peter Gray, the meeting adjourned to Monday, 30th March, 1868.

Fifth Ordinary Meeting, Monday, 30th March, 1868.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.
The following gentlemen were elected Associates, viz.:—

Francis Gustavus Paulus Neison, Jun.
Alfred Broughton Lamb.
William Beaman.

Mr. Henry William Manly read the Messenger Prize Essay, "On a Comparison of the values of Policies as found by means of the various Tables of Mortality and the different Methods of Valuation in use among Actuaries."

Thanks having been voted to Mr. Manly, the meeting adjourned to Monday, 27th April, 1868.

Sixth Ordinary Meeting, Monday, 27th April, 1868.

The President in the chair.

Read and confirmed the minutes of the last ordinary meeting.
The following gentlemen were elected, viz.:—

Fellow.
Henry Lake.

Associate.
David Shaw.

Mr. M. N. Adler, M.A., read a paper "On Insurance business in Germany."

Thanks having been voted to Mr. Adler, the meeting adjourned to Monday, 30th November, 1868.

The Twenty-first Annual General Meeting, Saturday, 6th June, 1868.

Samuel Brown, Esq., the President, in the Chair.

Mr. A. H. Bailey (Hon. Secretary) read the notice convening the meeting, and the following Report and Statement of Accounts:—

"The Council have to report that the number of members of the Institute on the 31st March last was 221; of whom 100 were Fellows, and 121 Associates. During the year, 29 new members have been elected, and 36 have left from various causes; but the diminution in the number is apparent rather than real, the list having been cleared of members whose subscriptions were in arrear. The amount of annual subscriptions received is greater than in the preceding year.

"The financial affairs of the Institute having increased in importance, the Council, acting upon a suggestion made by the Auditors, have adopted an improved method of keeping the accounts; and the results are now presented in a form which it is hoped will afford fuller information than formerly. It will be seen that the Annual subscriptions are more than sufficient for the ordinary expenditure; and that the assets, exclusive of the library, now amount to £1,480. 7*s.* 5*d.* Of this there are specifically appropriated, £281. 12*s.* 9*d.* to the Mortality Experience Inquiry, £220. 9*s.* 4*d.* to the Messenger Prize Fund, and £188. 18*s.* 5*d.* to the Hardy Memorial Fund. The remaining £789. 6*s.* 11*d.*, which includes £168 contributed by seven gentlemen who have compounded for their subscriptions, constitutes the General Fund, and shows an increase during the year of £100. 16*s.* 3*d.*

"The Council regret that the Mortality Experience investigation has been delayed by causes beyond their control. It was only on the 1st November, 1867, that the whole of the cards were received from the contributing Offices. Since that time unremitting attention has been given to the work. The laborious process has now been completed of combining in single cards the information furnished where more than one Policy had been issued on the same life; and it has been ascertained that the number of lives to be dealt with is upwards of 160,000. Of these, about 128,000 are Healthy Males, about 18,000 Healthy Females, and the remainder, lives on which extra premiums were charged. The amount of the contributions for this investigation is £537. 5*s*. 0*d*., which has been increased by interest to £557. 6*s*. 3*d*. Of this £275. 13*s*. 6*d*. had been expended to the 31st March last.

"The Messenger Prize for the best essay on the 'Comparison of the values of policies as determined by the use of different tables of mortality' has been awarded to Mr. H. W. Manly, of the London and Provincial Law Assurance Society. The Council are of opinion that the essay is not only highly creditable to the author, but also that it contains information which will throw considerable light on a subject which is not so well understood as its importance deserves.

"Several additions have been made to the Library, and the revision of the catalogue, mentioned in the last report, has been completed.

"The proceedings in Parliament during the past year affecting assurance interests have been as follows:—

The Policies of Assurance Act, 1867.
The Act to amend the Law relating to Sales of Reversions.
The Life Policies Nomination Bill.
The Friendly Societies Bill. And
The notice given by the Vice-President of the Board of Trade for an inquiry into Assurance Associations.

"The following papers have been read at the Sessional Meetings:—

'Report of the Sixth International Statistical Congress, held at Florence.' By the President.
'On an Instrument for furnishing the D numbers to four figures each in Joint Life Annuity Tables.' By Mr. Jardine Henry.
'On Briggs's Method of Interpolation.' Translation by Mr. Hill Williams.
'On the arrangement of Commutation Tables.' By Mr. James Chisholm.
'On the Values of Annuities Certain.' By Mr. Makeham.
'On the rate of interest in loans repayable by instalments.' By Mr. Peter Gray.
'The Messenger prize essay.' By Mr. Manly.
'On Insurance business in Germany.' By Mr. Adler.

"With regard to the examinations, the number of candidates for matriculation continues to increase, but only a small portion present themselves for final examination. Certificates of competency have been awarded to Mr. Mountcastle and Mr. Strachan. The Council have determined that in future each examination shall be divided into two parts occupying three hours each, instead of one continuous sitting occupying five hours, as at present.

"The Institute has now been in existence for a period of more than twenty years; and the Council cannot but congratulate the members on the success with which the objects contemplated by its establishment have been attained.

"(Signed) SAMUEL BROWN,
"*President.*

"12*th May*, 1868."

Income and Expenditure for the Year ending 31st March, 1868.

	£ s. d.	£ s. d.
Amount of Funds, 31st March, 1867, viz.:—		
Mortality Experience Fund	491 11 6	
„	224 15 3	
„	194 19 4	
„	688 10 8	
		1599 16 9
Annual Subscriptions, viz.:—		
69 Fellows, Town, at £3 3 0	217 7 0	
26 „ Country, „ 2 2 0	54 12 0	
80 Associates, Town, „ 2 2 0	168 0 0	
40 „ Country, „ 1 1 0	42 0 0	
		481 19 0
Subscription compounded		31 10 0
Certificate Fees		10 10 0
Dividends and Interest, viz.:—		
Mortality Experience Fund	19 14 9	
Messenger Legacy „	6 4 1	
Hardy Memorial „	5 17 6	
General „	14 7 2	
		46 3 6
Mortality Experience Fund, further contribution		25 0 0
		£2,194 19 3

	£ s. d.	£ s. d.
Rent	100 0 0	
Salaries	99 11 8	
Journal	85 13 6	
Printing and Stationery	54 10 0	
Ordinary Meetings	27 14 10	
Library	19 4 1	
Repairs	15 5 10	
General Expenses	35 10 0	
		437 9 11
xpenses		254 13 6
Manly		10 10 0
s purchased		11 18 5
Amount of Funds, 31st March, 1868, as below, viz.:—		1480 7 5
		£2,194 19 3

Balance Sheet, 31st March, 1868.

LIABILITIES.

	£ s. d.	£ s. d.
Mortality Experience Fund		281 12 9
Messenger Legacy Fund, viz.:—		
£211. 1*s.* 10*d.* Consols, cost	203 17 8	
Unappropriated dividends	16 11 8	
		220 9 4
Hardy Memorial Fund—		
£200 Consols, cost	179 14 6	
Unappropriated dividends	9 3 11	
		188 18 5
General Fund		789 6 11
		£1,480 7 5

ASSETS.

	£ s. d.	£ s. d.
Union Bank of London Deposit Account		470 6 3
Three per cent Consols, £900 Stock		823 19 3
Cash in hand, viz.:—		
London and Westminster Bank	164 14 3	
Petty Cash	8 15 8	
		173 9 11
Arrears of Subscriptions		12 12 0
		£1,480 7 5

Examined and found correct, 29th April, 1868.

(Signed) B. Newbatt, } *Auditors.*
(„) G. W. Berridge, }

The PRESIDENT—"Gentlemen, in moving 'That the Report of the Council and Balance Sheet be adopted, entered on the minutes, and printed in the *Journal*,' I need make but very few remarks, as it appears to me that on the whole you will consider the Report very satisfactory. (Hear, hear.) There is one point which I think deserves our attention, and that is the state of our funds. We do not profess to lay by money in order to accumulate funds for any purpose hereafter, but to spend our income from time to time in furthering the objects of the Institute, although, as everybody knows, it is a capital thing to have money in hand to fall back upon occasionally in time of difficulty, or for any special object for which it may be required. In any case, it gives a degree of respectability which is very satisfactory to every one. I may also remark with regard to the progress of the mortality experience which we are collecting, that it certainly is a considerable time since we first entered upon this investigation. The importance of the subject, however, must be one excuse for this apparent delay; and then, again, we have had to deal with the returns of many Offices who have sent in their experience at different times. It was not till recently—in November last—that we were able to get all the cards together, but since that period we have made considerable progress, and I am sanguine of seeing something like a report presented before the close of this year. (Hear, hear.) Whatever that report may be, I doubt not it will be very interesting to the members, because it will embrace the basis of those theories upon which the great institutions that we have the honour of being connected with so essentially depend. It is very satisfactory to us also to know that the course which the Institute has pursued in collecting the mortality experience of the Offices is now being followed by the Assurance Companies of Germany. I mentioned last year at the annual meeting of this Institute that a similar Institute had been formed of highly scientific men in Germany, combined too with the practical managers and heads of Companies, and that they were engaging themselves in what constitutes the important features of our own Institutes—that is to say, they were examining into the scientific questions and also into the practical ones which we are called upon to investigate. They are now occupied in collecting their mortality experience, and they have sent to us to obtain our forms, so as to endeavour to obtain their experience on a system uniform with our own. It is a pleasing thing for us to know that the lead we have taken is being so ably followed on the continent. (Hear, hear.) I need not remark on the state of the law with regard to insurance at the present time. Several subjects interesting to us have been introduced into the Houses of Parliament during the session, some of which appear to have fallen through. Whether in the course of next session Parliament may not have something important to occupy their attention without going into the subject of insurance, I need not try to predict. Lastly, I may say, the Report concludes with a phrase which shows that the Institute has really succeeded in the important objects for which it was founded. We see it not only in the admirable papers in the *Journal*, which is looked up to as a scientific journal in all parts of Europe, and which is largely circulated also in America; but we find it in the questions which have been so ably discussed during the past year; and, moreover, there has been no lack of interest or value in the contributions to its pages with which we have been favoured. (Hear, hear.) With these remarks, gentlemen, I beg leave to move, 'That the Report of the Council and the Balance Sheet be adopted and entered on the minutes, and printed in the *Journal*.' I shall now be happy to hear any remarks any gentleman may wish to make upon the subject."

MR. SPRAGUE—"I have much pleasure in seconding the adoption of the report, and in so doing shall be glad to be allowed to make a few remarks. Our President has spoken of our assets, remarking that there is no accumulation of funds; but I think that a society like this does not aim at the accumulation of funds. As regards the number of our members, we seem now to be almost stationary; and it would, of course, be gratifying to see an increase in the number of our members. Still we must feel that we have attained a position

of considerable influence and usefulness, although perhaps not so great as we at one time hoped. Having then reached the position we now occupy, if we can hold our ground, extend the circulation of our *Journal*, and continue to make our influence felt throughout the world, as it now is, I think we may rest very well satisfied, even though our funds and the number of our members should not show any increase. I do not despair of seeing, in process of time, a large increase in the number of our members; and I hope that some day, sooner or later, we shall find that this Institute secures the support, not only, as at present, of a large section of the insurance world, but of the whole body of the profession." (Hear, hear.)

Mr. Lodge asked whether in the experience that was being collected any notice would be taken of diseased lives? Would they be eliminated from the sound lives with a view to the formation of a table?

The President said that they were separated, but the tables were not formed at present: it would be necessary to do it, as soon as time permitted.

Mr. Sprague added that there were 128,000 healthy males, and these were being treated by themselves.

Mr. Wallis said the report noticed the fact that, whilst so many candidates presented themselves for the matriculation examination, there were but comparatively few who came up for the final examination, and he had observed the same thing for many years past. He questioned whether it would not be advisable to take some steps to prevent this in the future. Possibly the third year's examination was too discursive in its character. It seemed to him to necessitate a great deal of knowledge which could not be obtained by mere study, and which it was very difficult for an ordinary person to acquire. There were so many obstacles in the way of a candidate passing the final examination that it was almost out of the question for anyone to present himself for it. He believed there were many gentlemen connected with the Institute of sufficient mettle to compete with the difficulties; but from some cause or other they did not seem to care to undergo the examination.

Mr. Lodge suggested as a reason why so few came forward for the final examination that the gentlemen who had passed the first and second years' examinations often met with official appointments as secretaries and managers, and consequently they had some scruple about presenting themselves for the third year's examination, not liking to face the possibility of being "plucked." Candidates who had not received any official appointment would not be so likely to have any such scruple. He thought the fact that so many gentlemen who had passed the first and second years' examinations had been selected to fill responsible positions in different Offices reflected great credit upon the Institute. (Hear, hear.)

Mr. Wallis thought that anyone who had passed the first and second years' examinations was practically qualified to fulfil the ordinary duties of an actuary of an Assurance Company; but as there was so much that was discursive in the final examination, candidates did not care to offer themselves and run the chance of not passing, considering that if they succeeded they gained nothing but what appeared to him to be a barren honour.

The President remarked that candidates who passed the third year's examination at once took the title of "fellow," and were awarded a certificate of competency. (Hear, hear.) He might say on behalf of the Council that this question had been several times considered, and they had always decided to make no alteration that would in any way weaken the force of the examination, in order to enable candidates to pass it. (Hear, hear.) On the contrary, they had come to the conclusion that these discursive questions were necessary, ensuring as they did that none but gentlemen of the highest order of talent passed the examination. (Hear, hear.)

Mr. Bailey was very glad Mr. Wallis had taken this opportunity of bringing forward the subject of the examinations, upon which great differences of opinion existed. But he thought the honourable member had misunderstood the object of the paragraph in the report which referred to the examinations.

The intention of the three examinations was something of this kind: the first year's examination was to be a matriculation examination in so far as this, that it should show that the candidate possessed a reasonably good knowledge of elementary mathematics: the second year's examination was to be confined exclusively to the theory of the subject, and it was not considered that when a candidate had passed it he was competent to undertake the practical duties of an actuary: and the third year's examination was directed primarily to practical subjects, to test whether the candidate was able to apply the knowledge that the second year's examination showed he had acquired. (Hear, hear.) There had no doubt always been a difficulty as to what should constitute the subject of the final examination. It was quite true that some of the knowledge necessary to pass it was not to be acquired by books, but by actual official experience, and he thought that was inevitable. Take, as an instance, the value of a reversion, a common case—no study of books would tell a man what rates of interest that reversion would command in the market, and without some such knowledge a man was really not competent to value reversions, although he might be thoroughly acquainted with all the text books on the subject. Many attempts had been made to reduce the quality of the third year's examination, but these attempts had met with very little success. He was prepared to stand by a remark made by Mr. Jellicoe, who had always taken great interest in this question, that he had found in his own experience —and he (Mr. Bailey) could fully confirm him in this opinion—that the effect of laying down a variety of subjects was to induce young men to study political economy, statistics, finance, and to acquire a knowledge of a number of subjects of that kind which they did not previously possess, which was very beneficial to them, and which they would not have acquired but for this third year's examination. (Hear, hear.) His own feeling with regard to the matriculation examination was, that a great many gentlemen who were not members of the Institute came there to be examined in elementary mathematics, and had no intention of following the actuarial profession. Some of them were teachers of mathematics, and were more competent to teach the examiners than they to examine them. He thought that was a mistake, that it was not the function of this Institute to examine gentlemen for two guineas, who did not afterwards pay any Subscription. (Hear, hear.) In his opinion that went far to explain why so many came up to pass the elementary examination who did not come up a second time, and he thought that in this particular some amendment was required. (Hear, hear.)

Mr. Sprague asked to be permitted to express his full concurrence in what Mr. Bailey had said as to the third year's examination. There was a great deal in that examination that could not be learnt out of books, and he thought it was proper that it should be so, for the professional knowledge required in an actuary could be learnt from no books. Therefore, since the Institute gave certificates that certain persons were competent to act as actuaries, they did quite right to set questions which could only be answered by persons having a practical knowledge. It was quite true that many gentlemen passed the first year's examination who did not present themselves for the second, but that might happen from a variety of causes. When a young man had left the university, for instance, very often he did not know what line of life he would take up. Thinking that such an one had done well as an actuary, he might feel disposed to see if he could not succeed too, and with that view he would come up for the first year's examination. He (Mr. Sprague) thought it certainly an anomaly that the Institute should examine in elementary mathematics a gentleman who had taken a degree in mathematical honours at Cambridge or Oxford; but the way to meet that would be to excuse the first year's examination to gentlemen who could show by their degrees at Oxford or Cambridge that they had a sufficient knowledge of elementary mathematics, and allow them to present themselves in the first instance for the second year's examination.

The President observed that it was evidently to the credit of the Institute

that candidates should present themselves for the first year's examination in the expectation of getting something in return for it, as to a certain extent it proved that the certificate of the Institute was considered to be an honour. (Hear, hear.)

The report was then adopted unanimously.

A ballot was then taken for the election of President, Council, and officers. Mr. Neison, jun., and Mr. Stark having been appointed scrutineers, reported the election of the following gentlemen for the ensuing year:—

President.

SAMUEL BROWN.

Vice Presidents.

ALEXANDER GLEN FINLAISON.
THOMAS BOND SPRAGUE, M.A.
ROBERT TUCKER.
J. HILL WILLIAMS.

Council.

*MARCUS N. ADLER, M.A.
*ANDREW BADEN.
ARTHUR HUTCHESON BAILEY.
SAMUEL BROWN.
CHARLES JOHN BUNYON, M.A.
DAVID CHISHOLM.
GEORGE CUTCLIFFE.
ARCHIBALD DAY.
ALEXANDER GLEN FINLAISON.
ALEXANDER PEARSON FLETCHER.
WILLIAM JOHN HANCOCK.
AUGUSTUS HENDRIKS.
WILLIAM BARWICK HODGE.
CHARLES JELLICOE.
WILLIAM MATTHEW MAKEHAM.
*CHARLES TERRELL LEWIS.
FRANK McGEDY.
JAMES MEIKLE.
JOHN MESSENT.
EDWARD A. NEWTON, M.A.
WILLIAM P. PATTISON.
*ARTHUR PEARSON.
HENRY WILLIAM PORTER, B.A.
JOHN REDDISH.
HENRY AMBROSE SMITH.
*COL. JOHN THOS. SMITH.
THOMAS BOND SPRAGUE, M.A.
ROBERT TUCKER.
JOHN HILL WILLIAMS.
W. S. B. WOOLHOUSE.

Treasurer.

GEORGE CUTCLIFFE.

Honorary Secretaries.

ARTHUR H. BAILEY.
ARCHIBALD DAY.

The PRESIDENT, in returning thanks for the honour conferred upon him, the members of the Council, and the officers, said the great interest they had always felt in the Institute was clearly shown by the length of time with which most of them had been connected with it, some of them having been identified with it from the beginning. (Hear, hear.) They had all watched its progress with the greatest interest; and he was sure they must all feel gratified that it had been so useful in its objects and so successful in carrying out the original intentions of its founders. (Cheers.)

A vote of thanks was then given to the scrutineers.

Mr. Hargraves, Mr. Manly, and Mr. Mountcastle were appointed auditors or the ensuing year.

Mr. A. HENDRIKS proposed "That the best thanks of the meeting be given to the retiring President, Vice-Presidents, Council, and officers for their valuable services during the past year."

Mr. ANDREW BADEN seconded the resolution, which was at once agreed to.

The PRESIDENT returned thanks for this recognition of their services, and assured the meeting that they all felt the deepest interest in the Institute, and that the services they had rendered to it had been rendered with the most

* New Members.

cordial goodwill. (Hear, hear.) He trusted the Institute would continue to carry out as successfully as it had hitherto done the great objects for which it was established. They could not but feel that the confidence and support of all the members were essential to success. Whilst they had endeavoured to maintain the honour and reputation of the Institute he was sure that hereafter they would see a still more distinguished class rising up to exalt still higher its scientific character. (Hear, hear.)

The proceedings then terminated.

CORRESPONDENCE.

FIRE RE-INSURANCE LAW.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—Can you make room for the following suggestion?

Notwithstanding Mr. Bunyon's labours, to the value of which I bear most willing testimony, the code of principles ruling our Fire Insurance practice remains very much an unwritten law, and this remark applies with great force to the (Fire) re-insurance department, where, although all companies have consistently avoided the scandal of an appeal to Courts of Justice, yet differences are continually cropping up to be settled with more or less satisfaction, and, I fear, acrimony, by the usual reference to neutral managers.

Only a fortnight ago a leading Scotch Company sent round a Circular to the other associated Offices, asking their opinion on a disputed point, and I know of at least two more differences now pending.

My suggestion is that the decisions resulting from such differences being at present usually lost to all those not immediately concerned in them, their publication should be invited in the *Journal*—anonymously I suppose, although the names of the referees should certainly be given, and I commend the matter to your best consideration.

It is often a matter of regret expressed by the officers of Fire Companies that the *Journal* does not contain a fair proportion of information bearing on their special department; here is at least one branch waiting cultivation with a sufficiently practical bearing.

I am, Sir,

Your most obedient servant,

H. A. S.

London, 23*rd March*, 1868.

*** We shall have the greatest pleasure in inserting in this *Journal* not only any communications of the kind described by our correspondent, but any others that are of permanent interest to those of our supporters who are engaged in Fire Insurance business.—ED. *J. I. A.*

JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

Some Account of James Dodson, F.R.S. By A. De Morgan, Esq.

NO life has been written of the original projector of the Equitable Society, except in a column of the *Biographie Universelle* by M. Nicollet. Dodson's name was, and even still is, so familiar to the actuary, chiefly through the *Mathematical Repository,* and the impulse he gave to life-contingency problems, that this *Journal* is the proper place of deposit for what can be collected concerning him. The article above mentioned tells very little. He succeeded Hodgson [which should have been Robertson] in the chair of mathematics at *Christchurch Hospital* in 1756 [1755] and died November 23, 1757. He published the *Antilogarithmic Canon,* which others had contemplated [and executed too, but the manuscript was lost] and which he had the courage to execute up to a certain point [his table is the counterpart of Vlacq's largest direct table: five figures of argument and eleven of tabular result]. He could not *balance* the success of the ordinary tables: the writer doubts whether the table was ever used on the continent [he might have added, England: who *uses* either Vlacq or Dodson? Their tables are for help to other table-makers, and always were, though both of them intended more]. He published the *Calculator* in 1747, a collection of tables at the end of which [say in the proper

place in the middle] is an abridgment of the antilogarithmic table. But he is best known in England by his *Mathematical Repository*, and by his zeal for benevolent institutions [say his determination to found an assurance office to which himself should be admissible]. In his lectures at the school of Christ-church Hospital he gave the first idea of a company for life assurance, a plan afterwards executed by Edw. Rowe Mores [and others] under the name of the *Equitable Society* [I may safely contradict the statement that he lectured on life-assurance to the young men whom he was to instruct in mathematics and navigation].

James Dodson was my mother's father's father. All knowledge of him was completely cut off from his posterity by his leaving no near relation, no widow, and no child above fourteen years of age. I have, in several cases, found biographical inquiry arrested by similar circumstances. He seems to have had but two children, both sons. One, the elder, reared a large family, and must have, by this time, upwards of a hundred and fifty descendants, dead and alive: but never more than one male descendant of the name in my generation. So much for the efficacy of a large preponderance of daughters in preserving a surname. Of the other, nothing was ever remembered except that he "gave his brother much trouble." With all my inquiry, curious as I was to know all about this ancestor, I never obtained from his family more than three pieces of information: the date of his eldest son's birth; a copy of his treatise on book-keeping, which seems to have been preserved by his son, and which was given to me by one of his grandsons; a tradition that he was befriended on some occasion by the Duke of Manchester, who I have no doubt was a misnomer of the Earl of Macclesfield.

This paper is a case of the problem of constructing an unknown biography out of materials equally common to all mankind: and a sketch of a career may be given, as complete as many which are taken from contemporary record, and of much better evidence as to the separate facts than the unsupported statement of a casual writer. What may be done by one who takes the interest of a descendant in the matter is equally possible to be done by others: and a person who systematically collects all the biographical facts he meets with may find himself in a condition to give no small number of accounts, sufficient for literary purposes, of persons whose lives have been neglected.

James Dodson must have been born shortly before 1710: who he was, or from whence, I never found the slightest information.

From their long and close intimacy it must be suspected that he was a contemporary, perhaps a schoolfellow, of John Robertson (born 1712) the author of the *Navigation*, who in the history of Christ's Hospital is called the brother of Robertson the historian. But the following is more direct. In 1756, he found he could not assure in the Amicable Society, being over age: their limit was 45; and all accounts imply that he had but just passed the limit. He must have had some sort of liberal education; for his use of the Bernoullis, Euler, Ozanam, &c., shows that he read Latin and French. He must have been thrown on the world with some little command of money. He was able to spend unprofitable years in the construction of his antilogarithmic table, which he published on his own account in 1742: it was his first public appearance. A publisher's name (Wilcox) is joined with his own in the imprint: but we may be pretty sure that a folio of new tables at £1. 2*s*. 6*d*. (afterwards reduced to 12*s*.) by a young man quite unknown, would not find a publisher to take any risk. Those whom he mentions as his friends are Robertson, William Jones, of whom presently, and Labelye, who was, I believe, then building Westminster Bridge. Again, he had been, as we shall see, a pupil of De Moivre, who was at the top of the tree, and who must have been, at the time of Dodson's pupillage, very well remunerated, as one of the most famous of mathematicians, and Newton's particular friend. Between the *Canon* and the next work on his own account, he added a wife to his means of expenditure, which looks as if the money were not quite gone. He must have married soon after the publication of the *Canon*, for his first son was born in 1743. He was, I suppose, an amateur worker up to this time: for he is not called 'teacher of the mathematics' in the title of the *Canon*, though, had he been thus employed, the advertisement would have been a very good one: it first appears in 1747.

I have said he was a pupil of De Moivre. This is attested by Matthew Maty (M.D., afterwards Sec. R. S.) in his life, which is very little known, of his most particular friend De Moivre. Maty gives, as specimens of the pupils, Macclesfield, Cavendish, Stanhope, Martin Folkes, Fatio de Duillier, Scot, Daval, and Dodson. Of Lord Macclesfield I need say nothing; nor of Stanhope (the well known Lord Chesterfield), Folkes and Duillier. Cavendish was probably Lord Charles Cavendish, the father of the great chemist. Scot[t] was probably one of two fellows of the Royal Society of that name. Daval was a noted lawyer of a scientific turn, no doubt the Peter Daval who became Secretary

of the Royal Society in 1759. Those who look into the history of the time will find evidence of a *De Moivre clique,* kept together by intercourse with their old teacher, who lived until 1754, and by Maty's interest in the pupils of his old friend. Maty edited the *Journal Britannique,* a London publication in French, which expired shortly after De Moivre's death, living long enough to contain the biography mentioned, which was soon published separately. When any one of the pupils published a work, it was immediately favourably reviewed. When Sam Johnson's dictionary appeared, the review suppressed all about the celebrated letter to Lord Chesterfield, and hinted that Johnson should not have cast off the patron he himself had chosen at the beginning. So Johnson said of Maty, "He! the little black dog! I would throw him into the Thames:" from this we draw an inference which, in some very grave and dignified dictionary, will one day appear as "We have the testimony of the celebrated Dr. Johnson that Maty was short and dark: some take the great lexicographer as saying that he was of a surly temper, and not so much given to ablution as would in our time be held desirable; but we doubt if we can safely adopt this interpretation."

Various relations between the pupils are found. Lords Macclesfield and Chesterfield moved and seconded the second reading of the change of style; and Daval drew the bill. Dodson dedicated to his old teacher, and to the two peers; by whom he was also employed in surveying and accounts. I trace him through his writings as a private teacher, accountant, surveyor, &c., probably an answerer of actuary's cases, until 1755, when he gained what was for him a splendid rise in the world.

Charles II., who was a dabbler in science, and sometimes in a more creditable way than assisting at the joke of dissecting the body of an infant picked up about the palace,—and who really had that sense of the importance of navigation which an English Sovereign ought to have,—founded three *Royal* Institutions: the Royal Society, the Royal Observatory, and (1673) the Royal Mathematical School, attached to Christ's Hospital, for mathematics and navigation. The "New System of Mathematics" (2 vols. 4to. 1681) was written for this school by Jonas Moore, Master of the Ordnance, by whose advice it was founded: the course was left not quite finished, and Halley and Flamsteed took part in its completion. This school has always been distinct from the ordinary teaching of the Hospital, being especially devoted to navigation: and I have seen an elementary work announced as

intended for both the schools. At first the teachership was an office of very high consideration. When the Royal Society nominated Halley on the committee for keeping Newton to his work (the *Principia*) or as they phrased it, to "keep Mr. Newton in mind of his promise," the second member, who had a mere sinecure, would certainly be a person whose position was a guarantee for most respectful meaning on the part of the Society; especially considering the curious nature of the duty. Except in this instance I never heard of a scientific body extorting a promise that a book should be written, and appointing a committee to see that it was done. This second member was Mr. Paget, or Pagett, master of the Royal Mathematical School: and that he was selected for his position rather than his merits I infer from his carelessness as a teacher being notorious; he afterwards took to drinking, or perhaps we should say that his having taken to drinking afterwards became as notorious as his neglect of his duties. The post gradually declined in external notoriety, as the Royal School—which still exists—was more and more nearly absorbed into the Hospital. Very few of those who hear of the boys annually presenting their charts, &c., for the inspection of the Sovereign are aware that this privilege belongs to the Royal Mathematical School, and not to the Blue Coat School itself. It may be gathered from various circumstances that the post was, in 1755, no mean addition of station to the private teacher who had lived by all kind of odd jobs at "the Blue Legg, near to Bell Dock, Wapping." He gained it, as I suppose, by the influence of Lord Macclesfield, who was then President of the Royal Society: I thus interpret the imperfectly remembered tradition of a granddaughter, that he was befriended by the Duke of Manchester. He was admitted of the Royal Society Jan. 23, 1755, which was probably before his appointment to the teachership in the same year. This is fully confirmed by the third volume of the *Repository*. The preface is dated Jan. 23, 1755, which means that he had waited to date his preface until he could put F.R.S. after his name: a precedent for the Society, should it ever want one, that the *admission*, not the *election* (which had taken place a week before) gives the *literary* character. But he is not styled as of the R.M.S.: only "accomptant and teacher of the mathematics." The little point is to the following purpose. The Royal Society was somewhat exclusive during the last century, and rather averse to admit men in trade. But we must infer that Dodson was not elected because his new post made him grand enough, but that he might become

grand enough for the (probably) promised post. His friend Robertson, who preceded him, and whose position exactly resembled his own, had been F.R.S. since 1741: he held the post only about a year. And Hodgson, who came before Robertson, had been in the Society since 1703, five years before he gained the mastership. Accordingly, it seems to have been the rule to fill up the place from among the fellows of the Royal Society: but several of Dodson's early predecessors came into the school first, and into the Society shortly afterwards.

Dodson, thus comparatively enriched and established, wanted to insure his life, and found that the Amicable received no lives over 45. He accordingly set himself (1756) to found a new office; and thus became the projector of the Equitable Society, as presently described. Thomas Simpson was lecturing on the subject, with a view to a new office: Dodson called a meeting by advertisement, and formed a Committee. I find no trace of concert. I suspect that Simpson was looked on coldly by the De Moivre clique: many know the savage onslaught made by De Moivre on Simpson, though it seems the assailant afterwards cooled down. But it may be suspected that respect for the old man who represented the school of Newton, Leibnitz, the Bernoullis, &c., so long after they were gone, prevented much mention of Simpson, whom I do not find prominently cited by Dodson until after De Moivre's death, when he is spoken of in proper terms. A manuscript lecture of the period was lent to me many years ago, which showed no sign of being either by Simpson or by Dodson. Perhaps the plan was stirred in several quarters.

Dodson must have found his position very troublesome. His pupils were about twenty years of age: and the mixture of these men with the boys of the school led to all kinds of disturbance, beginning probably with interchange of chaffing and cuffing. But he did not enjoy it long; he died November 23, 1757. He leaves the character of a useful mathematician, inventive in application, but not in augmentation, of his science. He was eminently effective, and this until long after his death,—indeed, until 1820 at least—in attracting the attention of students of annuities and assurance to the problems connected with their subject. His term of public life was only fifteen years: and he was of a period in which the study of pure mathematics in England was at the lowest ebb. Had a man of thirty-two years old emerged from obscurity in the early time of Newton with such a folio as the *Canon*, no doubt the work of years, he would have been noticed and

encouraged: but nothing of this sort took place. To get an idea of our state at the time, say 1740–1760, take the names of all who were alive in Britain, no matter at what age, in any part of that period, and who can in any way be identified with pure mathematics. We have the remains of the old school, Berkeley, De Moivre, Halley, Jurin, Maclaurin, Robins, Stirling, William Jones and Braickenridge; a powerful list. To them we may add Thomas Simpson, Matthew Stewart, Walmesley, Waring, Robert Simson, Atwood, Hutton, Emerson, Horsley, Maseres, Playfair, Judge Wilson. Dodson, then, is one of the larger stars of his constellation: but the constellation not one of first-rate brilliancy. Reuben Burrow would have been added to my list, if he had published anything of sufficient note: but he appears in another way. Again, look forward to 1807, when we should see the crop of the seed-time just examined. In Mr. Walker's group—published six years ago—of fifty-one men of science of that day, the only two who are at all associated with pure mathematics are Leslie and Playfair.

An inquiry into the state of mathematical studies at Cambridge would probably confirm what I have said. Before such men as Waring, Paley, Milner, Vince, &c., gave strength to the system, I suspect that it was much debilitated. Taking the general results of *senior* wranglership as one test, there is little to speak of until the effect of those I have named began to be seen: and then we have such phenomena as three years which produced two bishops and a lawyer of celebrity, followed by five years which produced four judges. Of the dead period I have but one anecdote which I know to be true: it will look much like caricature. The senior wrangler of 176– was in 1825 still resident in his college, and of course very old. He recommended a young candidate for honours, in presence of one from whom I heard it, to be sure to attend particularly to quadratic equations: it was a quadratic, said he, which made me senior wrangler.

Any degree of celebrity, small or great, is not fairly established until detraction is proved: but this confirmation, as to Dodson, only turned up in our own day. The private diary of Reuben Burrow, a good mathematician, but eminently scurrilous and slanderous, is the place of deposit. For ample proof of this character, see the *English Cyclopædia* 'Tables,' and also *Notes and Queries*, Series I. vol. xii. p. 142 and Series III. vol. v. pp. 107, 215, 261, 303, 361. Burrow did not come into rivalry with Dodson, who is therefore let off cheaply: but poor Wales, against

whom Burrow was an unsuccessful competitor on more occasions than one, particularly for Dodson's old place in 1775, is, with another, "two of the most stupid and most dirty of all possible fools, rogues, and scoundrels," while Wales, he by himself he, is "not only the dirtiest scoundrel that God ever made, but the dirtiest rascal that he possibly could make. Amen." This is in the fly leaf of a book in my possession: my reader will not need me to tell him that Wales was an irreproachable man. From the diary it appears that this character is not entirely given on scientific grounds: for the wife of the said Wales is charged with having been the person who circulated the story that the said Burrow had given his own wife black eyes, a likely thing *per se*. The diary states that Wm. Jones, the father of the Indian Judge, so celebrated for his library and for his allowance of its use—the liberty of his study, Dodson calls it—as well as for his wide acquaintance with the mathematicians, was exceedingly rough and uncourteous: "Gardiner, the logarithm fellow, and Dodson, he used to treat like a couple of dogs." This is against all evidence of Jones's character: and I mention it first to note that Burrow calls Jones the Secretary of the Royal Society, which he never was; and gives Robertson—who was then clerk of the Society—as his informant; who must have known better, and who may safely be set down as never having said so. Probably Burrow confounded his man with Jezreel Jones, who was clerk of the Society, 1698–1713. William Jones was a Welchman, brought up in Wales: and a certain irascibility is held to belong to the national character. In that day, it must be remembered, the temperament of the races was much more pronounced than in our day, in which it would be easy to pick out and bring together an Englishman, a Welchman, a Scotchman, and an Irishman, of whom a fifth person, after hearing them talk for an hour, would be puzzled to say which was which. It may be held credible that Jones occasionally flew out: and his genial disposition, which led him to lay his treasures open to all, especially to the young aspirants whom he was so ready to advise and assist, probably had two warm sides, one at each end. But he had passed a life among his superiors both in station and in science, and all the probabilities of the case, as well as general evidence, are against his having had any reputation for habitual roughness. No name of the period has come down to us in a clearer atmosphere of respect and esteem. Burrow then gives the following account (Aug. 22, 1775).

"I had a good deal of talk with Mr. Robertson, and staid supper. He told me that Mr. Wm. Jones wrote that history of logarithms prefixed to Dodson's tables of the Anti-logarithmic Canon: that Dodson wrote such a confused and odd style that there was neither head nor tail in it, hardly; and that he himself drew up the examples. He also gave me the history of the Mathematical Repository, as follows. Mr. Robertson having taught General Conway mathematics (who was then only a colonel), after he was member of parliament he called on Mr. R. and told him that as his place in the House hindered his further attendance to mathematical subjects he should drop it, but at the same time he should be glad to have those papers which he had learnt copied over. Mr. R. not having time or inclination to do this himself applied to Dodson. Dodson employed one Ralph to copy them, but at the same time Dodson took a copy for himself (which by the bye was a dirty action). This Mr. R. did not know to a long time after, when, happening to think on the scheme of publishing a mathematical repository, the first volume of which was to contain a volume of algebraical questions, and the second geometrical, he proposed it to Dodson, who readily accepted the offer of joining with him. This Mr. Robertson mentioned to Mr. Jones, but Mr. Jones told him he was against the affair on account of Mr. R.'s probability of publishing some of the methods Jones had taught him, which he (Jones) might have thought of publishing afterwards himself. Mr. R. on this set the affair aside himself, but Dodson went on with it, and the greatest part of the questions in the first volume, at least 200 of the questions, were copied from Mr. Robertson's papers.

It will be worth while to follow up Mr. Burrow, because diary stories have been much relied on, and it will be instructive to point out what their value may be. I will therefore take one of a different kind, also derived from Robertson, upon which, as it happens, we are probably able to confront Burrow with Robertson himself. N.B. The blanks are not Burrow's.

He [Lord Macclesfield] married a ——, his family were in confusion, and when he died the —— ordered all his papers to be burnt but such as related to money matters, and Jones (*sic*) papers never was (*sic*) seen nor heard of more. Some think they were burnt among the rest, but Horsfall, of the Temple, who was one of those employed, says there were no such papers among those that were burnt. Others say that a number of papers were sent down to Shirborne Castle in his lifetime . . . . .

The two octavo volumes of Macclesfield Correspondence—which are but a small portion of the manuscripts now at Shirburn Castle—refute the tale of the burning. And now as to the character of the second Lady Macclesfield. Lord M. is described as having married, in 1757, 'Dorothy, daughter of — Nesbitt.' This short description probably indicates that her family was not of rank or note: but I can find nothing against her in the scandal of the time. Burrow must have mixed her up with *another* story,

which he probably *did* hear from Robertson, who left a slight written account of William Jones among his papers, which Hutton published. Robertson seems to have been one of those retailers of half-told stories who leave their hearers to fill up in their own way. Jones was the director of Lord Macclesfield's education until the young man travelled in France and Italy: and Robertson says "They tell a story of an Italian wedding, which caused great disturbance in Lord Macclesfield's family, but was compromised by Mr. Jones; which gave rise to a saying that Macclesfield was the making of Jones and Jones the making of Macclesfield." The compromise of a wedding was a thing which might have happened in those days, when the marriage-law* of England was the old law of Europe, which we now call the Scotch law. If the story have any foundation the young lord must have made some exchange of declarations in Italy, with a woman who followed him to England, and Jones may have been employed in buying her off. This seems somewhat supported by the haste with which a wife was found for the young man, who set out on his travels about 1720, and was married to his first wife in 1722. Probably Burrow has spoilt the point of the epigram by reversing the points: if Jones extricated the son, and the father afterwards gave him a good place, it would have been that Jones was the making of Macclesfield, and (then) Macclesfield was the making of Jones. But probably the reference is to some place given by the son, in addition to those already given by the father: Jones was certainly "made" long before 1720.

I now go on to what directly concerns Dodson, who says he got his questions out of mathematicians of the two centuries preceding, of whom he names twenty-one. Burrow says that more than 200 were exercises given by Robertson to General Conway, whom no one will believe to have mastered any 200 that can be pointed out. It is not credible that Dodson, himself a teacher, and a large importer of new algebra into a new subject, should have found it necessary to crib the simple equations, &c., of another teacher. Nor do I believe that Robertson told any such story of his friend past, present and future; especially to such a person as he knew Burrow to be. No doubt he told Burrow something: and Burrow had a power of inference not

* There was in England an inveterate popular belief, without any foundation in law, that the declarations which made a marriage must be made before a person in orders, English or Roman. There is a great deal of confusion on this subject, in great part arising from not remembering that the marriage by declaration before witnesses, which was *binding* both civilly and ecclesiastically, was held *irregular* by the Church, and made the parties subject to spiritual censures and penances; and also to some statutory penalties, which were seldom or never enforced.

given to all. It was one of those eccentricities of genius in private life—to use the phrase of a biographer—by which he was as much distinguished as by his *nihil quod tetigit non d—navit.* Lord Howe did not convoy the India fleet until they were out of (Burrow's) fear of the French: so it is laid down that "he and his brother are a couple of cowardly scoundrels, or else that they are bribed by the enemy." This was followed by what was perhaps the nearest approach Burrow's mind could make to *Domine, salvam fac patriam,* but worded thus—"What d——d stupidity this cursed nation of ours has fallen into!" Truly he is a person who tempts to digression.

Dodson, in the preface of the Canon, acknowledges much assistance in the drawing up of the explanation from Robertson himself, not from Jones. The part which is worth dwelling on is what relates to Jones. If Dodson wrote a fair common English, the whole falls to the ground. His printed writings show nothing either odd or confused: but he may have got somebody to write them *all.* He could hardly have kept a composer for his own private letters; and I subjoin one to Robertson, which came into the hands of Dr. Hutton, from whom it passed to Dr. Olinthus Gregory, at whose sale I bought it. The reader is to see whether the meaning comes at once or whether he must read a sentence twice before he understands it.

Sir. Being the other day turning over Mr. Simpson's new book, I took it in my head to try how much better his new approximations to the roots of equations were than those we commonly use, and find that his examples are packed, being such as our common operations will give to six or seven places the first substitution; which, with all his apparatus, he seldom exceeds above a figure or two. I determined therefore to reject his pretended improvement and stick to the old way in the work I am putting together for Mr. Knapton [what this was I do not know] and set about composing that part of it.

I believe you have found as well as I that these approximations are difficult to be worded so as that a person who cannot read algebra should readily understand and retain them [Dodson was very fond of expressing algebra in words, and did it with unusual precision and clearness]; but it has happened that in this revision of the subject I have, by a little cooking of the old equation, happened upon the following approximation to the root of any surd, which I give you in words that you may see how easy it will be to remember it.

The number whose root is required I call the surd power. And the nearest similar real power, whether greater or less, I call the rational power.

Multiply the rational power by the *index more one,* and to the result add the product of the surd power by the *index less one;* reserving the sum.

Also multiply the rational power by the *index less one*, and to the result add the product of the surd power by the *index more one;* reserving also this sum.

Then as the first mentioned sum is to the second, so is the root of the rational power to the root of the surd power.

I have sent you this in hopes it may come in time enough for the cube root in the arithmetical part of your navigation. And for that root it runs easier, thus.

To twice the rational cube add the surd cube, and to twice the surd cube, add the rational cube. Then as the first sum is to the second, so is the root of the rational cube to the root of the surd cube. (Please turn over.)

The investigation, being rather too long for a letter, I reserve till I see you unless you desire it further, when I will transcribe and sent it.

We have had a fortnight of very indifferent weather, but make shift to keep jogging on, and I am in great hopes the field-work may be finished before I am obliged to come to town: my next shall enclose the draft, which should have come now, but Sir Tho. is from home. I am, Sir, your obliged humble servant, J. DODSON.

Sept. 17, 1752, *by Act of Parl*[t] *style.*
[This was the fourth day of the new style.]

And now for a letter from William Jones, which I happen to possess; the man of influence and official station, who used small mathematicians like dogs; and who was the corrector of Dodson's style. So far as one letter can go, it clears him of both imputations. It is to Hodgson, Dodson's predecessor but one, and is addressed on the outside "To Mr. James Hodgson, at Christ's Hospital, London, these presents." Hodgson's life was a counterpart of Dodson's: he was a private teacher and writer who ended in the mastership of the Royal Mathematical School.

Honoured and beloved Sir. The Wednesday I came away I delivered the papers to your servant. It's my design to send them up in a little time, the calculations of problem (4) at large, so that everything may be evident to you as you proceed, without any trouble. I have altered the method from case (1) of Astronomic Problem (6) to case the (2) and hope to render it of more easy, universal and exact use. I will send one for the papers, and fairly insert problem (4) in writing among the others, and send them up to you without fail as soon as possible. I remain, most worthy Sir, your most obliged humble servant, W. JONES.

Wantage, June 17, 1731.

It is somewhat remarkable that so decided an instance of confused style should turn up, to set against the clearness of Dodson's writing. The reader asks how Jones could send up from Wantage the papers which he had left with Robertson's servant some Wednesday before: and he finds at last that "them" refers to papers spoken of afterwards.

Dodson's criticism upon Simpson's method refers only to its

utility as a means of approximation, and is just: but neither Dodson nor Simpson himself saw its beauty as a theorem. As it is never mentioned in modern works I give it, without demonstration, in modern symbols.

To approximate to the small root of an equation, proceed as follows. Let the equation be $c_0+c_1x+c_2x^2+\ldots\ldots=0$, and determine* D_1, D_2, &c., N_1, N_2, &c., from

$$c_0D_1+c_1=0,\quad c_0D_2+c_1D_1+c_2=0,\quad c_0D_3+c_1D_2+c_2D_1+c_3=0,$$
$$c_0N_1+c_2=0,\quad c_0N_2+c_1N_1+c_3=0,\quad c_0N_3+c_1N_2+c_2N_1+c_4=0,$$

and so on. Then

$$-\frac{c_0D_n}{c_1D_n-c_0N_n}$$

is the nearer to the root of the equation, the greater n is taken.

Dodson's share in the projection of the Equitable is first mentioned in general publication by Nichols (*Anecdotes*, vol. v., p. 400). But the following extracts, with which I was favoured by Mr. Arthur Morgan, contain the whole account.

In 1769 was circulated by the Directors a pamphlet entitled "A state of the Society for Equitable Assurances on Lives and Survivorships, and a state of facts from the year 1756 to the present time. Laid before the General Court the 28th of July, 1769, by the Court of Directors."

The following is an extract.

1756. In this year Mr. James Dodson, having been refused admission to the Amicable Society on account of his age, determined to form a new Society upon a plan of assurance on more equitable terms than those of the Amicable, which takes the same premium for all ages. Having communicated this plan to several persons, they proposed to join him therein, if the intended Society could be established by Charter. The number of persons which engaged in this design were at first 55, and before they procceded towards obtaining a charter, they set about providing a fund, and previous even to this consideration they held consultations about the plan of reimbursement and recompence that should be made to Mr. Dodson and themselves. Accordingly it was determined that 15*s.* should be paid by every person making assurance with the said Society; 5*s.* whereof should be paid to the said James Dodson for his life for his plans and trouble in planning the said Society, and making the necessary calculations; and the other 10*s.* were to go among the other persons [Raw beginners! primitive Christians! In our day this would be called *omission*, not *commission*: I never blushed for an ancestor until now.] The

* I use D and N because they enter in the demonstration as denominator and numerator. I suppose the D and N of our commutation tables were chosen by Griffith Davies from the part they play in $\frac{N}{D}$ the first of the results wanted, and the suggesting formula. But this never struck me until now; and perhaps never struck some of my readers.

application for a charter was conducted by Mr. Mores, and after three hearings before the Attorney and Solicitor General, to whom the petitioners were referred by his Majesty, a report was given against the petitioners. The petition having been presented at the Secretary of State's Office on the 16th or 17th of April, 1757, and referred to the Attorney and Solicitor General, who did not make their final report until the 28th of July, 1761. In the mean time, that is to say on the 23rd of November, 1757, Mr. Dodson died. The hopes of a Charter being at an end, the generality of the original subscribers dropt the scheme, in the prosecution of which £600 had already been expended. [In the deed of settlement provision is made for the repayment of this money]. Mr. Edw. Rowe Mores, however, and 16 more of the 55, resolved to persevere in establishing such a Society by deed, if it could not be done by Charter; and the present deed of settlement, of the 7th of September, 1762, was executed by every one of these 16 original Charter-fund proprietors. No table of calculations was procured till the 24th of January, 1764, and the Directors relying upon Mr. Mores for fixing every premium in the intermediate time. But at length such a table of lives was procured from the Executors of Mr. Dodson, and a resolution was put on the minutes for giving £300 to the children of Mr. Dodson as a recompence for the same.

In a statement published and signed by Rich. Glyn, J. Sylvester, Wm. Sclater, Edw. R. Mores, and Josiah Wallis, in reference to the Charter-fund, is found the following.

The subscribers admit that in the year 1756, Mr. Dodson, not being able to obtain admission into the Amicable Society on account of his age, conceived a design of forming a Society upon the principle laid down by the late Dr. Halley, in his observations on the Breslau bills of mortality, viz. that the price of insurance on lives ought to be regulated by the age of the person upon whose life the insurance should be made. And that he, Dodson, caused to be inserted in the public papers an advertisement bearing date the 28th of February, 1756, giving notice of a meeting intended to be holden on the 2nd of March then next following, and desiring at that meeting the company of such gentlemen as might be disposed to engage in such an undertaking. That they did accordingly meet upon the day appointed, and continued to meet weekly till the number amounted to about one hundred.

Mr. William Morgan, in his 'account of the Rise and Progress of the Equitable Society,' gives the account of the finish of Dodson's connexion with the Society, as follows.

Mr. Mosdell, who was stated to have been only an accountant, was appointed by the deed of settlement to be the first actuary, and on his death in December, 1764, [probably after six months trial, for the Equitable books show that the appointment is dated July 5, 1765,] Mr. [James] Dodson succeeded, who was the son of the excellent mathematician who computed the Society's tables, but without the mathematical learning of his father [he was then just twenty-one years old, and the appointment must have been an acknowledgment of the father's services]. Upon obtaining a place in the Custom House more suitable to his abilities, Mr. Dodson resigned in April, 1767, when Mr. John Edwards was chosen . . . .

I will now give a few words to each of Dodson's works. I find them all mentioned in Watt's *Bibliotheca,* that is, all which I have ever seen; and I never heard of any others. And the heading is one of those short accounts which often occur in Watt, which could not be mended in the same number of words. "Dodson, James, F.R.S., an ingenious and very industrious mathematician in London."

The Anti-logarithmic Canon. Being a Table of Numbers, Consisting of Eleven Places of Figures corresponding to all Logarithms under 100000. Whereby the Logarithm for any Number, or the Number for any Logarithm, each under Twelve Places of Figures, are readily found. With Precepts and Examples, showing some of the Uses of Logarithms, in facilitating the most difficult Operations in common Arithmetic, Cases of Interest, Annuities, Mensuration, &c. To which is prefix'd, An Introduction, Containing a short Account of Logarithms, and of the most considerable Improvements made, since their Invention, in the Manner of constructing them. By James Dodson. London: Printed for James Dodson, at the *Hand* and *Pen* in *Warwick-Lane;* and John Wilcox, at *Virgil's* Head, opposite the *New Church* in the *Strand.* 1742.

I should like to have a list of the authors who have shown their sense in the first words of the title of their first works: Dodson would find a place. The words "desiderandus videtur Canon *Anti-Logarithmicus*" were used by Wallis as far back as 1693. Young men very often think it is *original-like,* you know, to find their own phrases where good ones have been found by their foregoers. There is an appendix of five pages, not mentioned in the title. "Of *Decimal Notation,* and its Use in solving Questions, which consist of *Fractional Numbers* by Logarithms." The work is dedicated to Lord Stanhope (Chesterfield). It was reviewed in the *Works of the Learned* for September, 1742, in so terse and accurate a way, and so free from eulogium, that I have no doubt the author wrote the article.

There is a tangled story about the antilogarithmic Canon finished in manuscript by Warner and Pell. The utility of common slanderers lies not in what they produce, but in what they omit: as to all of which there is a strong presumption that no means of constructing a story existed. If there had been a rumour, even a surmise, afloat that Dodson had seen this manuscript, Burrow would have got hold of it, and would have left it that Dodson had cribbed the work out of William Jones's library, and had published it as his own. And nothing but a very cautious comparison will show that he had not the *opportunity* so to do. For Collins's papers, in the bulk, came into the hands of William

Jones, and were freely open to the crowd who had the liberty of his study: and Collins was certainly at one time the custodier of Warner's manuscript. No doubt a Canon with eleven hundred thousand computed figures, "elegant, in a large folio," would have been well known among the many mathematicians who haunted Jones's house and who knew what Wallis had written about it; and its surreptitious publication would have required the complicity of Jones and Robertson, at least, and the character of the transaction would have been known to many. But this is not all: it appears that Warner's manuscript, deposited with Collins to be restored on demand, actually did pass out of Collins's hands into those of Dr. Busby. It has never since been mentioned as seen. The authorities for the following collection of facts, Wallis, Pell, Thorndyke, and Collins are to be found in the *Latin* Algebra of Wallis (*Opera*, vol. ii., *Alg.* cap. xii.); the *Macclesfield Correspondence* (vol. ii., p. 197, 215, 219); and Halliwell's *Letters on Scientific Subjects* (Hist. Soc. Sci., pp. 80, 94, 95).

Dr. Pell informed Wallis that Warner, assisted by himself, had finished a *canon* not long after 1631: "about fifty years ago," says Wallis, which makes his writing to be near 1680, and very likely later. Wallis saw this canon, about 30 years before writing, say near 1650. In 1644, Pell, writing to Sir Chas. Cavendish, is in trouble about Warner's papers, the custodiers of which had become bankrupt, and he feared the papers had been or would be destroyed. We can only hope that poor Pell met all his troubles with as good heart as this one.

> In the mean time I am not a little afraid that all Mr. Warner's papers, and no small share of my labours therein, are seazed upon, and most unmathematically divided between the sequestrators and creditors, who (being not able to ballance the account where there appeare so many numbers, and much troubled at the sight of so many crosses and circles in the superstitious Algebra and that blacke arte of Geometry) will, no doubt, determine once in their lives to become figure casters, and so vote them all to be throwen into the fire, if some good body does not reprieve them for pye-bottoms, for which purposes you know analogicall numbers are incomparably apt, if they be accurately calculated.

The papers were found, and in 1652, we find them in the possession of Dr. Thorndyke, prebendary of Westminster, who as the trustee and holder of Warner's papers, among which the full canon and an abridgment are particularly specified, writes to Pell to urge publication of the whole, and seems to admit that Pell has the copyright: a note by Pell, endorsed on the letter, states that publication is abandoned on account of incompleteness, not of the

Canon, but of "the papers." No more appears until December, 1667, when we find the receipt given by Collins to Thorndyke, acknowledging the receipt of the Canon and other papers, to be restored on demand. In Sept., 1675, Collins writes to Tschirnhaus as follows:—

Between the years 1630 and 1640 Dr. Pell and one Mr. Warner, deceased, mentioned in Mersennus, agreed to make a table of antilogarithms, which were to be called Antilogarithmi Pellio-Warneriani; and accordingly such a table was computed, and left in the hands of Dr. Thorndyke, deceased, and cost Mr. Warner above 400 crowns the doing: as to the table itself it is a table of 99998 mean proportionals between an unit and 100,000, each to eleven places of figures, elegant, in a large folio....

Thorndyke was dead, and Collins does not say he had the table in his possession when he wrote: probably Thorndyke's executors found Collins's receipt and reclaimed the papers. Again, Collins, four years before, writing to James Gregory, March, 1671, gives the same account, as follows:—

One Mr. Warner, deceased, whose Optics you find mentioned in Mersennus, did, about 32 years since, spend above an hundred pounds for aid, and took great pains himself, with some assistance from Dr. Pell, to calculate a table to twelve places of figures of 100,000 continual proportionals, to wit, to find 99999 mean proportionals between an unit and 100,000. Such a large table, elegantly writ, remains in the hands of Dr. Thorndyke, a prebendary of Westminster; the construction and uses of it, with the tactions of circles rendered analytical, were lent to one Gibson, deceased, in anno 1650, author of a book entitled Syntaxis Math., after whose death all his papers were consumed to light tobacco." (*Maccl. Corr.* ii. 219.)

And again (p. 197, in a letter of which the date must be altered) "the tables I mentioned in Dr. Thorndyke's hands." So that the manuscript had gone back from Collins in 1671. It is passing strange that Collins, who was very well informed, and whose immense correspondence got him the name of the Attorney-General of the mathematics, should have been quite ignorant of Warner, Harriot's executor and the publisher of his very celebrated algebra, except as a person mentioned by Mersenne who, on like grounds, should have been the *Procureur-Général.*

Wallis, when he *wrote* his note, not far from 1680, to which he put a last paragraph after 1685, says that Pell—who must have known all about it—told him the papers were in the hands of Dr. Busby, of Westminster school, a very likely man to be the executor of the prebendary, and a very unlikely man to come by mathematical papers in any other way. When Pell made this communication to Wallis, he was meditating immediate publication, and his business was to ask Wallis to see the printing finished, in

case of his own death during the proceeding; to this Wallis assented. In 1755, Dr. Birch procured for the Royal Society some of Pell and Warner's papers from the trustees of Dr. Busby. The antilogarithmic canon—I mean the manuscript, to avoid all mistake; Dr. Busby himself was a canon, and probably an antilogarithmic one—might, or might not, have been among them. Rigaud inadvertently writes that Birch procured "four large boxes" of these papers for the Society: but Birch only says that the Pell and Warner papers were mixed with Busby's papers in four large boxes: if these boxes exist, the canon may be in them still. But it strikes me as most likely that Pell, a man of energy and impulse, after arranging with Wallis, obtained possession of the manuscript with intent to publish immediately, and that it was mislaid at his death. He was a "shiftless man," and shirtless too, sometimes: he often wanted pen and paper; he was in the King's Bench not long after his conversation with Wallis; and he died in poverty, and was buried at the cost of Dr. Busby.

It is clear that Dodson had no opportunity of seeing Warner's manuscript in the possession of William Jones or any one else that we know of. But it would be strange if there were none to suspect that he got at it amosgepotically (that is, somehow or other) and made fraudulent use of it. *A priori* wisdom will find difficulties in any other hypothesis. Why should Dodson, of all persons, meditate so large an undertaking, and why an antilogarithmic canon rather than anything else? He knew Wallis's account, which he quotes; and he might have seen Collins's letters in Jones's collection. What more easy than to suppose that he made a hunt for the manuscript? Suppose him to have once been a Westminster boy—he must have gone to school somewhere—and to have made use of his knowledge to gain access to Busby's boxes; what more is wanting? But though amusing myself with the love of evil which cannot help inventing all that is wanting to prove it, I am quite aware that it is open to anyone who can to trace the manuscript, and to examine the circumstances, in order to see whether—all apparent impossibility notwithstanding—Dodson found it and used it. It is quite certain that the fact of such a manuscript having existed must have been known in William Jones's circle: Wallis in print and Collins in the letters in Jones's library must have been referred to when Dodson published his Canon: and the acquisition of Busby's papers, in 1755, is presumption that the possibility of obtaining the manuscript was recognized; and not quite despaired of.

The only thing to be explained is the accordance of Warner and Dodson in extent of plan: five figures of argument and eleven of tabular result. The explanation is that both Warner and Dodson naturally aimed at making their tables coextensive with the great tables of Briggs and Vlacq. These had *eleven* figures; we now say ten: but the characteristic formerly counted as a place of the logarithm. Both Warner and Dodson judged correctly that their table would lose much of its working value for high purposes if, going above seven figures, it were anything less than the numerical counterpart of the great tables, which must be used with it.

Dodson very fairly quoted all he knew about Warner; that is, he gave the passage from Wallis's Algebra of 1693, in English. But, apparently dissatisfied with the translation which had appeared in two editions of Sherwin's logarithms, he translated anew, referring to Sherwin.

I will here mention that the correction of misprints found in the copies of the Canon are in most cases in Dodson's own handwriting. He followed this practice in more works than one.

1747. Octavo (half sheets). The Calculator: being, correct and necessary Tables for Computation. Adapted to *Science*, *Business*, and *Pleasure*. By James Dodson, *Accomptant, and Teacher of the Mathematics.* London: Printed for John Wilcox, at *Virgil's Head*, opposite the *New Church* in the *Strand*; and James Dodson, next Door to the *Blue Legg*, near *Bell-Dock*, *Wapping*. M.DCC.XLVII.

This work is dedicated to William Jones. Some copies have another title page, also of 1747, in which Wilcox alone is mentioned in the imprint. This means that Wilcox took the risk off Dodson's hands within the year; and thenceforward we no more find him publishing on his own account.

With the exception of heavy calculators, to whom the Canon is occasionally useful—Benjamin Gompertz, for instance, who told me forty years ago he was always wanting it—this table is worth three of the Canon to anybody. Whoever can catch a copy should keep it. The table of binomial coefficients, up to the 34th power, is very useful. So is the table of specific gravities. The medley of coins, measures, regular solids and polygons, roots, logarithms, common, hyperbolic, logistic, trigonometrical, &c., interest, annuities, &c. &c., though not extensive, are great friends at a pinch. For a single book to travel with, and a good chance for anything that can be wanted, I know only Mr. Willich's table which can compare with it. But Dodson's two or three words to each head in the preliminary index enable the user to find his table in a

moment. The only additional friend mentioned is John Ellicot, F.R.S., the (in that day) celebrated watchmaker.

1747–48, 1753, 1755, 12mo. The three volumes of the Mathematical Repository, by James Dodsòn, Accountant and Teacher of the Mathematics. The full titles will recal the contents: they describe the volumes as

(i.) Containing analytical Solutions of near five hundred questions, mostly selected from scarce and valuable authors. Designed as examples to Maclaurin's and other elementary books of algebra; and to conduct beginners to the more difficult properties of numbers.

(ii.) Containing algebraical solutions of a great number of problems, in several branches of the mathematics. I. Indetermined questions, solved generally, by an elegant method communicated by Mr. De Moivre. II. Many curious questions relating to chances and lotteries. III. A great number of questions concerning annuities for lives, and their reversións; wherein that doctrine is illustrated in a multitude of interesting cases, with numeral examples, and rules in words at length, for those who are unacquainted with the elements of these sciences, &c.

(iii.) Containing analytical solutions of a great number of the most difficult problems, relating to annuities, reversions, survivorships, insurances, and leases depèndent on lives; in which it has been endeavoured to exhaust the subject.

All is 'printed for John Nourse, at the Lamb, opposite Katherine Street in the Strand.' The dedications are to De Moivre, David Papillon, F.R.S., and Lord Macclesfield and the Council of the Royal Society. There was a second edition of the first volume in 1775; I am not aware of any other editions of the remaining volumes. I should think there were none, for the remaining stock of the work was locked up by some of the incidents of trade, and was let out about 35 years ago, when the market was suddenly supplied with uncut copies.

1750, 4to. The Accountant, or the method of book-keeping, deduced from clear principles, and illustrated by a variety of examples. By James Dodson, Teacher of the Mathematics. London printed for J. Nourse at the Lamb opposite Katherine-Street in the Strand.

This book is dedicated to Lord Macclesfield, whose accounts Dodson seems to have been employed in, and who, it is hinted, desired that double entry should be applied to the business of an estate and of a farm. The work also applies it to retail trade, a thing till then unexemplified: and the shoemaker's trade is chosen on account of the variety of his transactions. This book is excessively scarce: the copy in the Museum and my own being the only ones I ever heard of.

1751. 8vo.. In this year Dodson published an enlarged edition of Wingate's Arithmetic. The preface is dated April 4, 1751.

It was reprinted several times. I have only seen the edition of 1760. Wingate is the best of the old writers, greatly superior to Cocker (or rather Hawkins): and Dodson's are the best editions of Wingate, according to Watt—and myself. Wingate and Cocker were the two household-gods of arithmetic. In 1750, Arthur Murphy introduced their names upon the stage in 'the Apprentice,' Wingate as an old merchant who is constantly recommending Cocker: and I believe that this is the way in which Cocker became a bye-word; I can find nothing earlier.

> 4to. (pp. 18). An account of the Methods used to describe lines, on Dr. Halley's Chart of the Terraqeous Globe; showing the Variation of the Magnetic Needle about the Year 1756, in all the known Seas; their Application and Use in correcting the Longitude at Sea; with some Occasional Observations relating thereto. By William Mountaine and James Dodson, Fellows of the Royal Society. London: Printed for W. and J. Mount, T. Page and Son, on *Tower Hill.* 1758.

Of Mountaine I only know that he was one of the founders of the Equitable, and that he was Dodson's executor. Watt calls this tract a folio, and gives it a first edition in 1718. The truth, as appears by the tract itself, is that in 1744 the two collected observations from the Admiralty, the India and African companies, and private communications. On these data they published a chart in 1745, which I have never seen. By this chart Dodson must have been known as having paid attention to matters connected with navigation, a circumstance which may have facilitated his appointment to the R. M. School.

To the preceding list must be added three papers in the *Philosophical Transactions;* 1752, p. 333, on the improvement of the bills of mortality; 1754, p. 487, on annuities and survivorships; 1753, p. 273, on logarithmic series. The second and third papers are written to show how to dispense with the use of fluxions, which all the mathematicians who could were very apt to intrude into every part of algebra above the merest elements. This practice did much harm: the packing up of all the difficulties of series into the abbreviations of the differential calculus was a fearful drawback on the rigour of the science. It is only in our own day that mathematicians have become alive to the danger of all sorts and conditions of interminable series. Here is an instance for the reader of the mathematical part of this *Journal.* Take the series

$$\frac{4-3x}{1.2}x+\frac{16-15x}{3.4}x^3+\frac{36-35x}{5.6}x^5+\dots$$

This is certainly convergent when x is $<$ or $=1$. When $x=1$, it seems to be $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\ldots.$ or $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots.$, or hyp. log 2. But it is not: what is it, then?

The algebra of annuities, &c., was put into working form by De Moivre, Simpson, and Dodson, who gained the necessary restraint upon themselves by having been occupied in the actual practice of the subject. It is almost a rule that a writer on any mixed mathematical subject who has not been actually engaged in *mixture* overdoes the mathematical part: I do not mean that he introduces mathematics where it ought not to be—this he may or may not do—but that he makes too much of mathematics where some ought to be. De Waring, one of the most useful algebraical discoverers of the century, made a great failure in an attempt to write on the subject: and as the history of his tract is peculiarly matter for this *Journal,* I will end with it.

The book was called 'On the principles of translating algebraic quantities into probable relations and annuities, &c. By E. Waring,' Cambridge, 1792, 8vo. (pp. 59). It would have sold well if the implied title-promise had been kept: it is not every one who can translate algebraic quantities into an annuity, or into a probable relation with the chance of a reversionary legacy. As it was, no book ever fell more dead from the press: it is not mentioned by any of Waring's biographers before 1815. Some notice of it was taken in the first edition of Hutton's *Mathematical Dictionary,* vol. ii. p. 276, which induced Mr. Baily to write to Hutton for information. Hutton answered that he had never seen nor heard of the tract; that the account in which it appeared was furnished by Waring himself, whom he took to be good authority for a work of his own: that it certainly was not one of the pamphlets which passed between Waring and Powell during the contest for the professorship; but that he had found a "referment" to it in Wood's Algebra. Baily accordingly wrote to Dr. Wood, who in answer gave the title, and offered to lend his own copy (Sept. 5, 1808). The offer was accepted; for on the 15th Baily wrote to Hutton a short account of the work, which he described in stronger terms than he afterwards used in his book on assurances (Pref. p. xx.), and in which I quite agree with him. "Certainly he has thrown no new light on the subject. His problems (if a string of detached observations are worthy of that name) are quite

elementary, and his loose and illogical [he meant *immethodical**] method of treating them adds neither grace nor dignity to the subject. The very title of the book betrays the inaccuracy of his style." Baily afterwards picked up a copy for himself. When I came to look after this book, about 1835, I could find no mention of it: and I asked Mr. Baily to lend it to me. He could not find it; and I ventured to express a suspicion that he had mistaken the author's name. Whereupon he produced† what he knew where to find at once, the bookseller's receipt, which stated name and title. It turned up when his books were arranged for sale, and I bought it. Some time afterwards I found that the library of Queen's College, Cambridge (which was not Waring's College), contained some half dozen copies. A few of these were, upon representation of the state of the case, presented to other libraries, I forget which: probably the Royal Society or the British Museum will now possess the book.

I have never had so strongly impressed upon me the littleness of the period preceding the accession of Geo. III. We do not make much boast of its collective literature, and yet it was the day of Mansfield, Fielding, Sam. Johnson, David Hume, Sterne, Gray, Garrick, Blair, Hor. Walpole, Smollett, Robertson, Adam Smith, Blackstone, Joshua Reynolds. In applied science there was no great strength: but in pure mathematics we have little more than the remnants of a stronger period: some good names, but far too few to count as a school, belong especially to the time. Its historical masterpiece is the *Biogr. Philosoph.* of Benjamin Martin

* In reply to a suggestion whether *un*methodical would not be the preferable word, Mr. De Morgan writes:—"I made the word *immethodical*, upon the old analogy. *Un* is Saxon; and properly belongs to Saxon words, as *unaware*, *unbeaten*. *In* and *Im* are for Latin; though certainly the Saxon has intruded, as in *ungovernable*, *unsophisticated*, *uncommon*, &c. But the great bulk of our Latin words still keep *im* or *in*, according to the consonant which follows, as *imperceptible*, *immense*, *innocent*, and a crowd of others. On looking for *immethodical* in a little sixpenny Johnson of the stalls,—there it is. I generally consider the foreign dictionaries as good authorities as to English words: and in the French, German, and Italian which I keep at hand, I find the word in all. I find capricious cases; as *in*terminable and *un*terminated, *indeterminate* and *undetermined* (of which the mathematicians have availed themselves). Also *insatiate* and *unsatiated*. The rule seems to be that when the Saxon *ed* is at the end, the Saxon *un* shall be at the beginning; and Latin, Latin. This may be called the *sandwich* rule, if it be a rule. The end of it is that any one may do as he pleases, which is the glory of English."—Ed. *J. I. A.*

† Francis Baily was a paragon of method: he practised and enforced. I found him one day in the act of finishing a note, which he showed me; it was before the time of *prepaid* letters. One of those tradesmen who, when a customer is as good as the bank, persist in making a banker of him during convenience, would not send in his bill. The note ran as follows:—"(No. 1). Sir,—I beg you will oblige me by sending in your account forthwith. Yours, F. Baily. P.S. This notice will be repeated once a week until it is complied with." No. 2 was not wanted: the tradesman declined to grant His Majesty an annuity of 8*s.* 8*d.*, payable weekly.

(1761), a work of unmatched inutility. And yet good biography had commenced in force with the *Biogr. Brit.* in 1747. The total absence of historical effort encouraged the learned vicar of Twickenham, George Costard, to give to his work on the globes, (1767), full of every kind of miscellaneous historical statement, the title 'History of Astronomy.' All my reading has led me to suspect that the doubts and dangers of the disputed right to the Crown, which lasted from the rising of 1715 to that of 1745, produced a paralysing effect upon the intellectual energies of the country.

On the Application of Bonuses to limit either the term of an Assurance or the number of payments to be made under it: being Excerpts from a paper read before the Actuarial Society of Edinburgh, on 5th December, 1867. By James R. Macfadyen, *of the City of Glasgow Life Assurance Company.*

THE application of bonus to the hastening of the time fixed by the original contract for the payment of a life assurance policy, and the using of it to extinguish the premiums payable during the latter years of life, are systems, which, partly from their growing favor with the public, and consequent adoption by many offices, and still more from the fact, that the subject is by no means exhausted, notwithstanding various able articles that have appeared in this *Journal*, seem to me to render apology unnecessary for again treating of it.

The papers to which I allude as having already taken up the matter, are two articles by Mr. Sprague in the sixth volume, embracing both branches of the subject, and a more recent letter by Mr. Marr, (vol. xiii. p. 246) referring to the first of these two systems alone.

Though, as we shall hereafter see, these systems of applying bonus are very closely allied in character, and they have thus been coupled in this paper, it will be more convenient in the first place to confine ourselves to that in which the bonus is applied to limit the *term* of insurance only.

The problem usually arising in this case is to find the term of the Endowment Assurance into which a given cash bonus will convert an ordinary assurance. Mr. Sprague solves this problem by making an equation at the age the change is effected between ^1b, the cash bonus then allocated to the policyholder, which is the

benefit he relinquishes, and the difference between the costs of an Endowment Assurance for the unknown term, and an ordinary assurance for the whole of life, together with the values of all the future premiums after the maturing of the Endowment Assurance, which are the benefits given in return. From this equation, n, the unknown term of the Endowment Assurance can be found, and by treating this result by a similar method, the age can be found at which a second cash bonus will cause the assurance to be paid, and so with any other cash bonus that may be allocated.

Mr. Marr arrives at his formula in a different manner. He founds on the well known equation ($V_{x|y}$ representing the value of a policy at age $(x+y)$ which has been opened at age x)

$$\frac{V_{x|y}}{1+a_{x+y}}+\varpi_x=\varpi_{x+y}.$$

By making ϖ_{x+y} represent the Endowment Assurance premium for the unknown term $(n-y)$, and $1+a_{x+y}$, the annuity-due for the same period, and adding to the left-hand member the further benefit ${}^1\mathbf{b}$ given up by the assured, we have an equation formed, where n will appear as the unknown term of a known temporary annuity, and therefore may be readily found by a table of such annuities. By turning Mr. Marr's result into the D and N values, it can of course be reduced to the same expression as that of Mr. Sprague; and if, in either of the formulæ, the equation be manipulated so as to have ${}^1\mathbf{b}$ standing alone on one side of the expression, we have, n being given, the means of solving the converse problem —*i.e.* given the age at which the policy is to be payable, required to find what cash bonus is necessary for the purpose. Or if instead of ${}^1\mathbf{b}$ be substituted any other form in which the bonus may be declared, say, ${}^1\mathrm{B}\frac{\mathrm{M}_{x+y}}{\mathrm{D}_{x+y}}$, where ${}^1\mathrm{B}$ represents the reversionary bonus, we have the means of solving either of the preceding problems with reversionary instead of cash bonuses.

There is however a fourth method of working questions of this description, which has been alluded to, though not entered upon, in the letter of Mr. Marr already referred to, and it is to this method we shall now direct our attention.

Let $(x+y)$ be the age at first division of profit, ${}^1\mathbf{b}$, ${}^2\mathbf{b}$, ${}^3\mathbf{b}$, &c. the cash bonuses declared at ages $x+y$, $x+2y$, $x+3y$, &c., y thus being the investigation period, ${}^1\mathrm{B}$, ${}^2\mathrm{B}$, ${}^3\mathrm{B}$, &c., the corresponding reversionary bonuses, in which case ${}^1\mathbf{b}={}^1\mathrm{B}\frac{\mathrm{M}_{x+y}}{\mathrm{D}_{x+y}}$, ${}^2\mathbf{b}={}^2\mathrm{B}\frac{\mathrm{M}_{x+2y}}{\mathrm{D}_{x+2y}}$,

and so on. Let ϖ_x be the *pure* premium for £1 at age of entry, x; and let n be the term of the endowment assurance, or the quantity to be found.

Then, at first period of conversion, or at age $(x+y)$ the cash bonus ${}^1\mathbf{b}$ is employed to purchase an Endowment payable at the unknown age $(x+n)$.

When that time is reached by the assured, the amount of endowment then due, together with the surrender value of his original policy, is the sum that is at his credit with the Assurance Company. But, by hypothesis, he is now entitled to receive the sum assured by his policy;—

∴ the amount of endowment then due, + the surrender value of his policy, = the sum contained in that policy = 1. From this equation we can readily find n.

Should a second bonus ${}^2\mathbf{b}$ be allocated at age $(x+2y)$, then the amount of endowment payable at the new unknown age, that could be purchased at age $(x+2y)$, for a sum equal to ${}^2\mathbf{b}$
+ the surrender value of the endowment payable at $(x+n)$, for which ${}^1\mathbf{b}$ had been given
+ the surrender value, at the new unknown age, of the original policy
= 1, as before; and thus the new unknown age could be found.

Perhaps it will be more clear if the formulæ are given. Thus—

$$\text{Value of policy} + \begin{array}{c}\text{value of Endowments}\\ \text{purchased by } \mathbf{b}\end{array} = \text{Sum assured.}$$

or
$$A_{x+n} - \varpi_x(1+a_{x+n}) + {}^1\mathbf{b}\frac{D_{x+y}}{D_{x+n}} + {}^2\mathbf{b}\frac{D_{x+2y}}{D_{x+n}} + \&\text{c.} = 1;$$

whence,

$$1 - d(1+a_{x+n}) - \varpi_x(1+a_{x+n}) + \frac{{}^1\mathbf{b}D_{x+y} + {}^2\mathbf{b}D_{x+2y} + \&\text{c.}}{D_{x+n}} = 1,$$

and finally,

$$D_{x+n}(1+a_{x+n}) = \frac{{}^1\mathbf{b}D_{x+y} + {}^2\mathbf{b}D_{x+2y} + \&\text{c.}}{d+\varpi_x} \qquad (1)^*$$

An expression from which n can be found, whether the rates of interest and mortality assumed for the various parts of the expression be the same or not.

It will be well to point out here of what parts this formula is made up, as on the face of it, it does not clearly show, and as this must be considered in making practical use of it. The D's are calculated by the table by which the endowments are sold, the d

* In the above formulæ, the amount of Assurance is supposed to be £1. If any other amount be wished to be employed, say S, we shall have to substitute for $d+\varpi_x$, wherever it occurs, $S(d+\varpi_x)$.

and the a_{x+n} at the rates at which the policy is valued, and the ϖ_x is the pure premium originally paid.

Should the Endowments be calculated by the same Table of Interest and Mortality as the policy is valued by, (1) will become

$$N_{x+n-1}=\frac{{}^1\mathbf{b}D_{x+y}+{}^2\mathbf{b}D_{x+2y}+\&c.}{\varpi_x+d} \qquad (2)$$

This is the expression given by Mr. Sprague; and in using his formula, it ought to be remembered that this assumption is made in it.

If in (2) the bonus is assumed to be reversionary and is valued by the same table as the policy and endowments, we have

$$N_{x+n-1}=\frac{{}^1BM_{x+y}+{}^2BM_{x+2y}+\&c.}{\varpi_x+d} \qquad (3)$$

And should still further the amounts of the reversionary bonuses declared be the same

$$N_{x+n-1}=\frac{B(M_{x+y}+M_{x+2y}+M_{x+3y}+\&c.)}{\varpi_x+d} \qquad (4)$$

From one or other of these expressions can be found n, $\mathbf{b}$, or B, whatever be the combinations of interest and mortality introduced—the only supposition made in (1), from which the others are drawn, being, that at every investigation, each endowment is sold and valued by the same table.

There are various other forms in which these equations can be considered. Thus for $\frac{1}{\varpi_x+d}$ can be substituted (should ϖ_x and d be at same rate of interest) $1+a_x$; or, should the investigation be annual, and ${}^1\mathbf{b}={}^2\mathbf{b}=\&c.$, (2) becomes

$$N_{x+n-1}=\mathbf{b}(1+a_x)(N_{x+y-1}-N_{x+\theta y}).$$

If again, as is more common, ${}^1B={}^2B=\&c.$ and the investigation is annual, (3) becomes

$$N_{x+n-1}=B(1+a_x)(R_{x+y}-R_{x+(\theta+1)y}).$$

θ being the number of the investigation immediately preceding the payment of the policy.

It will be seen from this method of solving the problem, that there is really no change in the policy, and that so far as it is concerned, it is only the surrender value of it that is paid at age $(x+n)$. The assured simply enters into a new contract with the office, by sinking his bonuses in the purchase of endowments, while the policy is in the same state as it was at entry. This principle is of the greatest importance, and will be a guide through all the various intricacies that may arise in studying the question;

and as we shall find, it holds equally good, when the bonus is applied to limit the number of payments.

It will be observed that we have taken ϖ_x as the *pure* premium at age x. If we had used the *loaded* premium, the value of the policy would be less by an annuity-due of this loading; and thus the whole fund to provide profit and pay expenses beyond age $(x+n)$, would be anticipated and pressed into the calculation for the purpose of diminishing the value of the policy. Now the assurance runs off the books of the Company at age $(x+n)$, and it would be unjust to charge the policyholder for profits he will never receive, and to debit him with expenses he will never occasion; so the *pure* premiums ought to be used.

Neither in these formulæ, nor in those of Messrs. Sprague and Marr, does the policyholder get any credit for that portion of profit accruing between $(x+\theta y)$ and $(x+n)$—and this too, although during that period he is usually called on to pay a premium which is specially designed to produce a surplus to be returned. No doubt this is an injustice, and strictly speaking, the policyholder, if the Office be one declaring intermediate bonus between investigations, should have this intermediate bonus handed over to him at age $(x+n)$, along with the amount originally assured. We are inclined to think, however, that the more general practice of Offices allowing this application of bonus, is to make no return whatever to the policyholder under this head.

Another question which it may be well to consider here, is the value of these formulæ for practical computations. A glance will show that, in this respect, the finding of n will be a matter more or less tedious, according to the number of different tables of interest and mortality on which the various parts of the equation are calculated. If these be few, the formula will be correspondingly simple, and *vice versâ*. If even the most complicated of all, (1), be employed, the operation is not a very laborious one. From Mr. Chisholm's "Commutation Tables" a table of $\lambda D_{x+n}(1+a_{x+n})$ for the Carlisle Table, can be very rapidly formed, the logarithms of the factors being there given; and should it be merely wished to find what n will become by the application of a single bonus only, it will be unnecessary to find the number corresponding to this logarithm, as the form of the other member of the equation is peculiarly adapted for the use of logarithms, and it is as easy to find n from a comparison of the logarithms as from the numbers. And the same remark may be made of the other formulæ given, as Mr. Chisholm's work supplies the logarithm of N_{x+n-1}, or, as it is there termed, N_{x+n}, for the Carlisle Table.

The use of $1+a_x$ for $\dfrac{1}{\varpi_x+d}$, when it is admissible, will be an obvious advantage.*

As, after the next section of the present paper, any further researches we may make, will be equally applicable to assurances, whether limited in term or in number of premiums, we shall now proceed to investigate the formula for finding that age at which premiums may be made to cease by the application of bonuses. Mr. Sprague, in the articles before referred to, has taken up this subject also. The method he employs to find n is this:—

The benefit the Office surrenders is the payment of premiums after age $(x+n)$, and for this the policyholder gives his cash bonus ${}^1\mathbf{b}$; which produces an equation from which n can be found; and similarly, when a second bonus is allocated, the new age can be found.

There is, however, another method of finding n, somewhat analagous to that investigated already in the limiting of the term of assurance, but of which we have not observed any previous notice taken; and it is this we shall now examine.

Let P_x be the premium *actually paid*, let $(x+n)$ here mean the age *on* and after which no further payments are to be made, and let the other symbols be as previously.

Then at age $(x+n)$ the Company foregoes all future premiums, that is, gives up a benefit of $P_x(1+a_{x+n})$. To meet this, at successive declarations of profit, the policyholder pays the Office sums ${}^1\mathbf{b}$, ${}^2\mathbf{b}$, &c., which are allowed to accumulate to meet the single payment at $(x+n)$, requisite to redeem the premiums, that is, $P_x(1+a_{x+n})$—or, in other words, as in the first form of applying bonus, the policyholder purchases with ${}^1\mathbf{b}$, ${}^2\mathbf{b}$, &c., endowments payable at age $(x+n)$. Therefore

$$ {}^1\mathbf{b}\frac{D_{x+y}}{D_{x+n}}+{}^2\mathbf{b}\frac{D_{x+2y}}{D_{x+n}}+\ \ldots\ {}^\theta\mathbf{b}\frac{D_{x+\theta y}}{D_{x+n}}=P_x(1+a_{x+n}), $$

therefore

$$ \frac{{}^1\mathbf{b}D_{x+y}+{}^2\mathbf{b}D_{x+2y}+\ \ldots\ {}^\theta\mathbf{b}D_{x+\theta y}}{P_x}=D_{x+n}(1+a_{x+n}) \qquad (5) $$

whence n can be found.

* Since writing the above, Mr. Ralph P. Hardy has kindly put at my service, a method he employs to find n, when one bonus only is introduced, and both policy and Endowment are valued at same rates.

In (2), divide by D_{x+y}, and we have

$$ \frac{N_{x+n-1}}{D_{x+y}}=\frac{{}^1\mathbf{b}}{d+\varpi_x} $$

and, thus, we have the unknown member of this equation, take the form of an annuity deferred n years, of which the amount is known, as also the present age of the life. Whence n can be readily found, from a Table of such annuities.

Here, as in the previous case, the D's are calculated by the table of interest and mortality at which the policyholder purchases his endowments, and the a_{x+n} is founded on the table the Office uses in estimating the worth of the future premiums.

Should these be the same, (1) becomes

$$\frac{{}^1\mathbf{b}D_{x+y} + {}^2\mathbf{b}D_{x+2y} + \ldots {}^\theta\mathbf{b}D_{x+\theta y}}{P_x} = N_{x+n-1} \qquad (6)$$

which is the expression of Mr. Sprague, so that we have to repeat here the caution about using his formula without taking note of its limitations; the deferred annuity employed in it, containing both the endowments and the value of future premiums at age $(x+n)$, and those assumed at the same rates. It is an assumption, however, that practically is of little moment to attend to, as these rates will almost invariably be taken as the same.

If in (6) we wish to use the reversionary bonuses, it becomes, should they also be valued by the same table as the premiums are redeemed at, and the endowments are sold by,

$$\frac{{}^1BM_{x+y} + {}^2BM_{x+2y} + \ldots {}^\theta BM_{x+\theta y}}{P_x} = N_{x+n-1} \qquad (7)$$

and if $y=1$, and ${}^1\mathbf{b} = {}^2\mathbf{b} = {}^3\mathbf{b} = \&c.$, then (6) becomes

$$\frac{\mathbf{b}}{P_x}(N_{x+y-1} - N_{x+\theta y}) = N_{x+n-1}.$$

Or if $y=1$, and ${}^1B = {}^2B = \&c.$, then (7) becomes

$$\frac{B}{P_x}(R_{x+y} - R_{x+\overline{\theta+1}y}) = N_{x+n-1}.$$

From one or other of these formulæ can be found expressions for n, $\mathbf{b}$, or B, for any possible combination of interest and mortality rates, with the sole exception of the case when any one of the Endowments is sold or valued by a different table from the others, a form which, alike from the extreme improbability of its occurrence, and from the cumbrousness of the resulting formula, we do not feel called on now to consider.

In the application of bonus to limiting the term of assurance, we used ϖ_x the pure premium; but, in this case it must be the actual premium paid, as after age $(x+n)$ the policy will still continue on the books of the Company, draw profits, and cause expense. The single item of the loading that might possibly be omitted from P_x is perhaps that part of it which goes in commission, and that only on the supposition that no allowance is made to the agent for the sudden cutting off of his fountain of supplies. We

are here touching on a point that may some day cause trouble in the actual working, both of this scheme, and of that we first investigated.

We shall now proceed to make some comparison between the results of applying profit in either of the above mentioned methods, and those that occur by using a more ordinary system; and for this purpose we shall take the most common of all, viz., that in which the cash profits allocated are applied to purchase reversionary sums. For the sake of brevity we shall consider (ϵ) as the symbol introduced when either of the two schemes that form the subject of this paper is being treated of, and (η) as that employed when the applying of profits to purchase reversionary sums is meant.

Then let us imagine two persons of the age x, effecting policies of the same amount, the premiums in both cases being paid annually; and in each case at ages $x+y$, $x+2y$, &c., let cash profits, ${}^1\mathbf{b}$, ${}^2\mathbf{b}$, &c., be allotted to the policies. Suppose further, that the one (ϵ) applies these profits, as they are allotted, to limit either the term of his policy or the number of premiums payable under it, and the other (η) buys reversionary sums, ^{1}B, ^{2}B, &c., with his, how do the two policies compare at, say age $(x+t)$? Then up to the moment when the first bonus is allocated, they are exactly the same, but ever afterwards the value of policy (ϵ) is greater than that of (η),—a point which will tell strongly in favour of (ϵ), when we consider the practical working of the various methods of distributing profit. The reason of the difference in the values is evident. For (ϵ), by using his bonuses to purchase endowments, exposed the m, (*i.e.* the bonuses) to a risk which (η) does not, and, of course, in calculating the value of his policy, he will be entitled to the worth of that risk.* Thus, throwing it into symbols, if t be not greater than n, the term at the end of which the policy becomes payable, or the premiums are to cease (ϵ). If $V_{x|t}$ = the value of the original policy at present age, and if $(x+\theta y)$ be the age at which the bonus immediately preceding age $(x+t)$ is declared—then

$$\epsilon-\eta=\left\{V_{x|t}+{}^1\mathbf{b}\frac{D_{x+y}}{D_{x+n}}\frac{D_{x+n}}{D_{x+t}}+{}^2\mathbf{b}\frac{D_{x+2y}}{D_{x+n}}\frac{D_{x+n}}{D_{x+t}}+\ldots{}^{\theta}\mathbf{b}\frac{D_{x+\theta y}}{D_{x+n}}\frac{D_{x+n}}{D_{x+t}}\right\}$$
$$-\left\{V_{x|t}+{}^1\mathbf{b}\frac{D_{x+y}}{M_{x+y}}\frac{M_{x+t}}{D_{x+t}}+{}^2\mathbf{b}\frac{D_{x+2y}}{M_{x+2y}}\frac{M_{x+t}}{D_{x+t}}+\ldots{}^{\theta}\mathbf{b}\frac{D_{x+\theta y}}{M_{x+\theta y}}\frac{M_{x+t}}{D_{x+t}}\right\}$$

* It will perhaps make this clearer to say that (η) exposes the Office to an additional *current* risk,—that of having to pay the reversionary bonus—but (ϵ) exposes the Office to no such risk; and the values of the bonuses at the outset being equal, the outstanding liability of the Office, after the expiration of any assigned interval, is therefore greater in the case of (ϵ) than of (η).—ED. *J. I. A.*

is the difference between the values of (ϵ) and (η),

$$= \frac{1}{D_{x+t}} \left\{ ({}^1\mathbf{b}D_{x+y} + {}^2\mathbf{b}D_{x+2y} + \ldots {}^\theta\mathbf{b}D_{x+\theta y}) - M_{x+t}\left({}^1\mathbf{b}\frac{D_{x+y}}{M_{x+y}} + {}^2\mathbf{b}\frac{D_{x+2y}}{M_{x+2y}} + \ldots {}^\theta\mathbf{b}\frac{D_{x+\theta y}}{M_{x+\theta y}} \right) \right\}$$

But since (by hypothesis) $t > \theta y$, $\therefore M_{x+t} < M_{x+\theta y}$, and, *a fortiori*, $M_{x+t} < M_{x+y}$, $< M_{x+2y}$, &c.; and therefore (ϵ) is greater than (η): that is, the values of policies are greater when the bonuses are applied in either of the modes under consideration, than when reversionary sums are purchased with them.

The extent of the excess can be found by reducing our expression further. Thus it becomes

$${}^1\mathbf{b}\left(1 - \frac{M_{x+t}}{M_{x+y}}\right)\frac{D_{x+y}}{D_{x+t}} + {}^2\mathbf{b}\left(1 - \frac{M_{x+t}}{M_{x+2y}}\right)\frac{D_{x+2y}}{D_{x+t}} + \ldots {}^\theta\mathbf{b}\left(1 - \frac{M_{x+t}}{M_{x+\theta y}}\right)\frac{D_{x+\theta y}}{D_{x+t}},$$

or

$${}^1\mathbf{b}\frac{D_{x+y}}{M_{x+y}}\left(\frac{M_{x+y} - M_{x+t}}{D_{x+y}}\right)\frac{D_{x+y}}{D_{x+t}} + {}^2\mathbf{b}\frac{D_{x+2y}}{M_{x+2y}}\left(\frac{M_{x+2y} - M_{x+t}}{D_{x+2y}}\right)\frac{D_{x+2y}}{D_{x+t}} + \ldots {}^\theta\mathbf{b}\frac{D_{x+\theta y}}{M_{x+\theta y}}\left(\frac{M_{x+\theta y} - M_{x+t}}{D_{x+\theta y}}\right)\frac{D_{x+\theta y}}{D_{x+t}}.$$

That is, (ϵ) exceeds (η) by the assurance of the reversionary sum that **b** can purchase at each allocation of profit, for the period from that allocation till the date of enquiry, improved at interest and mortality rates for that period.

We have said that t must not exceed n; but we only did so because if the first scheme be adopted, the policy at age $(x+n)$ terminates, and the comparison ceases. If, however, the second form be adopted, as the payment of premiums only ceases, there is no reason why t should be limited to any age under the difference between $(x+y)$ and the oldest age in the tables. The proof just given, as it was quite independent of the value of t, will suffice to embrace this case also.

Therefore whether (ϵ) stands for bonus applied to limiting the term of an assurance, or the number of payments made under it, from the time of the first declaration of profit, till the time the policy runs off the books of the Company, the value of an assurance under scheme (ϵ), will exceed that of one under similar conditions, but of which the profits are applied to purchase reversionary sums (η), by a quantity for which we have just found the expression.

We have compared (ϵ) and (η) with regard to the values, and we shall now endeavour to resolve the question, whether, if the

rates of mortality and interest turn out differently from those founded on, the profit, if any, will be greater or less on system (ϵ), or on system (η); and here, in order to distinguish between expected and actual rates, we shall put an accent over the symbols referring to the latter.

Let, as before, $(x+t)$ be the age at which the examination takes place, and the other symbols have the same meaning as in the last problem; then, since, as we have already shown, the original policies under systems (ϵ) and (η) are still in the same state, and as the expression we wish to find is not one for the *amount* of the profit by either system, but simply a formula for its *difference*, we may reject altogether, as common to both, the profit made under the original policies, and narrow our investigations to a consideration of the effects produced on the cash bonuses allotted to the policies by systems (ϵ) and (η), subject to such conditions of mortality and interest as we have supposed. We shall also reject consideration of the intermediate bonus, as its cash value is all we can deal with, and it cannot be considered as applied in any way diverse in the one case from the other.

Then, under system (ϵ), at age $(x+t)$, since ${}^1\mathbf{b}$ allocated at age $(x+y)$, was invested in the purchase of an Endowment payable at, say, age $(x+s)$, of which the amount would thus be ${}^1\mathbf{b}\dfrac{D_{x+y}}{D_{x+s}}$; and the value of this at age $(x+2y)$ would be ${}^1\mathbf{b}\dfrac{D_{x+y}}{D_{x+s}}\dfrac{D_{x+s}}{D_{x+2y}}$; and this is again invested at age $(x+2y)$ in the purchase of an Endowment payable at, say, $(x+o)$; which is again valued and invested at $(x+3y)$; and so on; till at age $(x+t)$, the value of the Endowment is,

$$
{}^1\mathbf{b}\frac{D_{x+y}}{D_{x+s}}\frac{D_{x+s}}{D_{x+2y}}\frac{D_{x+2y}}{D_{x+0}}\cdots\frac{D_{x+\theta y}}{D_{x+n}}\frac{D_{x+n}}{D_{x+t}}, \text{ or } {}^1\mathbf{b}\frac{D_{x+y}}{D_{x+t}}.
$$

Then the expression for the profit on (ϵ) will be

$$
{}^1\mathbf{b}\frac{D'_{x+y}}{D'_{x+t}}-{}^1\mathbf{b}\frac{D_{x+y}}{D_{x+t}}-\left\{{}^1\mathbf{b}\frac{D'_{x+y}}{D'_{x+\theta y}}-{}^1\mathbf{b}\frac{D_{x+y}}{D_{x+\theta y}}\right\}\frac{D'_{x+\theta y}}{D'_{x+t}}.
$$

This is the expression for the profit on (ϵ) for the period between $(x+\theta y)$ and $(x+t)$; the surplus up to age $(x+\theta y)$ being distributed.

It will be observed that we value the endowment at the rates founded on, not on those realised, so as not to anticipate the future profit on it.

Similarly, the expression for (η) will be

$$
{}^1\mathbf{b}\frac{D'_{x+y}}{D'_{x+t}}-{}^1\mathbf{b}\frac{D_{x+y}}{M_{x+y}}\left\{\frac{M'_{x+y}-M'_{x+t}}{D'_{x+t}}+\frac{M_{x+t}}{D_{x+t}}\right\}
$$
$$
-\left\{{}^1\mathbf{b}\frac{D'_{x+y}}{D'_{x+\theta y}}-{}^1\mathbf{b}\frac{D_{x+y}}{M_{x+y}}\left(\frac{M'_{x+y}-M'_{x+\theta y}}{D'_{x+\theta y}}+\frac{M_{x+\theta y}}{D_{x+\theta y}}\right)\frac{D'_{x+\theta y}}{D'_{x+t}}\right\}
$$

By multiplying out, (ϵ) can be reduced to

$$
-{}^1\mathbf{b}\frac{D_{x+y}}{D_{x+t}}+{}^1\mathbf{b}\frac{D_{x+y}}{D_{x+\theta y}}\frac{D'_{x+\theta y}}{D'_{x+t}};
$$

and similarly, (η) can be reduced to

$$
{}^1\mathbf{b}\frac{D_{x+y}}{M_{x+y}}\frac{M'_{x+t}}{D'_{x+t}}-{}^1\mathbf{b}\frac{D_{x+y}}{M_{x+y}}\frac{M_{x+t}}{D_{x+t}}-{}^1\mathbf{b}\frac{D_{x+y}}{M_{x+y}}\frac{M'_{x+\theta y}}{D'_{x+t}}
$$
$$
+{}^1\mathbf{b}\frac{D_{x+y}}{M_{x+y}}\frac{M_{x+\theta y}}{D_{x+\theta y}}\frac{D'_{x+\theta y}}{D'_{x+t}}.
$$

Dividing by ${}^1\mathbf{b}\dfrac{D_{x+y}}{M_{x+y}}$, we have the difference between (ϵ) and (η), or $(\epsilon)\sim(\eta)$,

$$
{}^1\mathbf{b}\frac{D_{x+y}}{M_{x+y}}\left[\left\{-\frac{M_{x+y}}{D_{x+t}}+\frac{M_{x+y}}{D_{x+\theta y}}\frac{D'_{x+\theta y}}{D'_{x+t}}\right\}\sim\left\{\frac{M'_{x+t}}{D'_{x+t}}-\frac{M_{x+t}}{D_{x+t}}-\frac{M'_{x+\theta y}}{D'_{x+t}}+\frac{M_{x+\theta y}}{D_{x+\theta y}}\frac{D'_{x+\theta y}}{D'_{x+t}}\right\}\right]
$$
$$
={}^1\mathbf{b}\frac{D_{x+y}}{M_{x+y}}\left[\left\{\frac{M'_{x+\theta y}}{D'_{x+t}}-\frac{M'_{x+t}}{D'_{x+t}}\right\}\sim\left\{\frac{M_{x+y}}{D_{x+t}}-\frac{M_{x+t}}{D_{x+t}}-\frac{M_{x+y}}{D_{x+\theta y}}\frac{D'_{x+\theta y}}{D'_{x+t}}+\frac{M_{x+\theta y}}{D_{x+\theta y}}\frac{D'_{x+\theta y}}{D'_{x+t}}\right\}\right]
$$
$$
={}^1\mathbf{b}\frac{D_{x+y}}{M_{x+y}}\left[\left\{\frac{M'_{x+\theta y}-M'_{x+t}}{D'_{x+t}}\right\}\sim\left\{\frac{M_{x+y}-M_{x+t}}{D_{x+t}}-\frac{M_{x+y}-M_{x+\theta y}}{D_{x+\theta y}}\frac{D'_{x+\theta y}}{D'_{x+t}}\right\}\right]
$$
$$
=\frac{{}^1\mathbf{b}}{A_{x+y}}\left[A'_{\frac{x+\theta y}{t-\theta y|}}\frac{D'_{x+\theta y}}{D'_{x+t}}\sim\left\{A_{\frac{x+y}{t-y|}}\frac{D_{x+y}}{D_{x+t}}-A_{\frac{x+y}{(\theta-1)y|}}\frac{D_{x+y}}{D_{x+\theta y}}\frac{D'_{x+\theta y}}{D'_{x+t}}\right\}\right]
$$

and making the requisite changes for ${}^2\mathbf{b}$, ${}^3\mathbf{b}\ldots{}^\theta\mathbf{b}$, we have the expression for the total difference of profit between (ϵ) and (η),

$$
(8)\left\{\begin{array}{l}
\dfrac{{}^1\mathbf{b}}{A_{x+y}}\left[A'_{\frac{x+\theta y}{t-\theta y|}}\dfrac{D'_{x+\theta y}}{D'_{x+t}}\sim\left\{A_{\frac{x+y}{t-y|}}\dfrac{D_{x+y}}{D_{x+t}}-A_{\frac{x+y}{(\theta-1)y|}}\dfrac{D_{x+y}}{D_{x+\theta y}}\dfrac{D'_{x+\theta y}}{D'_{x+t}}\right\}\right]\\
+\dfrac{{}^2\mathbf{b}}{A_{x+2y}}\left[A'_{\frac{x+\theta y}{t-\theta y|}}\dfrac{D'_{x+\theta y}}{D'_{x+t}}\sim\left\{A_{\frac{x+2y}{t-2y|}}\dfrac{D_{x+2y}}{D_{x+t}}-A_{\frac{x+2y}{(\theta-2)y|}}\dfrac{D_{x+2y}}{D_{x+\theta y}}\dfrac{D'_{x+\theta y}}{D'_{x+t}}\right\}\right]\\
+\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\\
+\dfrac{{}^\theta\mathbf{b}}{A_{x+\theta y}}\left[A'_{\frac{x+\theta y}{t-\theta y|}}\dfrac{D'_{x+\theta y}}{D'_{x+t}}\sim\left\{A_{\frac{x+\theta y}{t-\theta y|}}\dfrac{D_{x+\theta y}}{D_{x+t}}\right\}\right]
\end{array}\right.
$$

Should $^{1}\mathbf{b}, {}^{2}\mathbf{b}, \ldots {}^{\theta}\mathbf{b}$, be constantly increasing, so that the reversionary bonus B, is always the same, then this expression becomes

$$(9) \qquad \mathrm{B}\left[\frac{\theta(\mathrm{M}'_{x+\theta y}-\mathrm{M}'_{x+t})}{\mathrm{D}'_{x+t}} \sim \left\{\frac{\mathrm{M}_{x+y}+\mathrm{M}_{x+2y}+\ldots\mathrm{M}_{x+\theta y}-\theta\mathrm{M}_{x+t}}{\mathrm{D}_{x+t}} - \frac{\mathrm{M}_{x+y}+\ldots\mathrm{M}_{x+(\theta-1)y}-(\theta-1)\mathrm{M}_{x+\theta y}}{\mathrm{D}_{x+\theta y}}\cdot\frac{\mathrm{D}'_{x+\theta y}}{\mathrm{D}'_{x+t}}\right\}\right]$$

and should the division of profit be annual, that is, if $y=1$, this last expression becomes

$$(10) \qquad \mathrm{B}\left[\frac{\theta(\mathrm{M}'_{x+\theta}-\mathrm{M}'_{x+t})}{\mathrm{D}'_{x+t}} \sim \left\{\frac{\mathrm{R}_{x+1}-\mathrm{R}_{x+\theta+1}-\theta\mathrm{M}_{x+t}}{\mathrm{D}_{x+t}} - \frac{\mathrm{R}_{x+1}-\mathrm{R}_{x+\theta}-(\theta-1)\mathrm{M}_{x+\theta}}{\mathrm{D}_{x+\theta}}\cdot\frac{\mathrm{D}'_{x+\theta}}{\mathrm{D}'_{x+t}}\right\}\right]$$

Having now obtained $(\epsilon)\sim(\eta)$, let us try to discover whether (ϵ) or (η) is the greater; and for simplicity, let us assume $t=2y$; that is, a bonus of $^{1}\mathbf{b}$ being declared on each policy at age $(x+y)$, with which in the one case (ϵ) an endowment payable at $(x+n)$ is bought, and in the other (η) a reversionary sum; what is the difference of the profit in the two cases, at age $(x+2y)$?

Making the necessary substitutions in (1), we have

$$^{1}\mathbf{b}\,\frac{\mathrm{D}_{x+y}}{\mathrm{M}_{x+y}}\left[\mathrm{A}'_{x+y\,\overline{y|}}\,\frac{\mathrm{D}'_{x+y}}{\mathrm{D}'_{x+2y}} \sim \mathrm{A}_{x+y\,\overline{y|}}\,\frac{\mathrm{D}_{x+y}}{\mathrm{D}_{x+2y}}\right]$$

—an expression symmetrical enough.

If the rates of mortality and interest actually realised, be precisely the same as those founded on, then the dashes will disappear: (ϵ) will equal (η), and $(\epsilon)\sim(\eta)$ become nil;—thus proving that an office acting under system (ϵ), loses as much by the premature death of a policyholder, as an office acting on system (η), even though it pays less. We should hardly call attention to this, were it not a popular argument in support of such schemes of dividing profit as those now treated of. It needed no formula, however, to show this, as it will be at once seen, that whatever benefit was gained by the less payments under the death claims, was discounted in the sale of the endowments. In fact, the same thing was shown, when it was proved that the values of such policies were greater than by methods when the assured grasped at less and ran less risk.

When the experienced mortality and interest are different from the expected, (*i.e.* those by which the benefits are sold), (ϵ) does not necessarily equal (η).

We shall first assume the experienced and the anticipated interest the same, but the actual mortality more favourable than the expected. Then, in the expression,

$$ {}^{1}\mathrm{b}\frac{\mathrm{D}_{x+y}}{\mathrm{M}_{x+y}}\left[\overset{(\epsilon)}{\mathrm{A}'_{\overline{x+y}\,\rceil y}\frac{\mathrm{D}'_{x+y}}{\mathrm{D}'_{x+2y}}}\sim\overset{(\eta)}{\mathrm{A}_{\overline{x+y}\,\rceil y}\frac{\mathrm{D}_{x+y}}{\mathrm{D}_{x+2y}}}\right] $$

$$ \mathrm{A}'_{\overline{x+y}\,\rceil y}<\mathrm{A}_{\overline{x+y}\,\rceil y} \text{ and } \frac{\mathrm{D}'_{x+y}}{\mathrm{D}'_{x+2y}}<\frac{\mathrm{D}_{x+y}}{\mathrm{D}_{x+2y}} $$

and therefore (ϵ) is less than (η).

This is also evident, when we remember that a mortality lighter than anticipated will cause the endowments to be sold at a loss, but the reversionary sums at a profit.

If, on the other hand, the mortality be heavier than provided for by the tables, (ϵ) will exceed (η), and the endowments will be sold at a profit, and the reversions at a loss.

Next, should we assume the mortality rates, actual and anticipated, the same, but the interest realised greater than that assumed, then,

$$ \mathrm{A}'_{\overline{x+y}\,\rceil y}<\mathrm{A}_{\overline{x+y}\,\rceil y} \text{ but } \frac{\mathrm{D}'_{x+y}}{\mathrm{D}'_{x+2y}}>\frac{\mathrm{D}_{x+y}}{\mathrm{D}_{x+2y}}; $$

so that in order to find whether $(\epsilon)-(\eta)$ is a positive quantity, we shall have to analyse the expression still further.

Leaving out the common multiplier (a positive factor), $(\epsilon)-(\eta)$ is

$$ \frac{(l_{x+y}-l_{x+y+1})v'+(l_{x+y+1}-l_{x+y+2})v'^{2}+\ldots(l_{x+2y-1}-l_{x+2y})v'^{y}}{l_{x+y}}\cdot\frac{l_{x+y}}{l_{x+2y}v'^{y}} $$

$$ -\frac{(l_{x+y}-l_{x+y+1})v+(l_{x+y+1}-l_{x+y+2})v^{2}+\ldots(l_{x+2y-1}-l_{x+2y})v^{y}}{l_{x+y}}\cdot\frac{l_{x+y}}{l_{x+2y}v^{y}} $$

Divide by $\frac{l_{x+y}}{l_{x+2y}}$, and multiply by $v'^{y}v^{y}$, (positive quantities, and therefore having no effect on the sign of the expression), and we have

$$ v^{y}\{(l_{x+y}-l_{x+y+1})v'+\ldots(l_{x+2y-1}-l_{x+2y})v'^{y}\} $$
$$ -v'^{y}\{(l_{x+y}-l_{x+y+1})v+\ldots(l_{x+2y-1}-l_{x+2y})v^{y}\} $$

which is the same as

$$ (v^{y}v'-v'^{y}v)(l_{x+y}-l_{x+y+1})+(v^{y}v'^{2}-v'^{y}v^{2})(l_{x+y+1}-l_{x+y+2})\ldots $$
$$ +(v^{y}v'^{y}-v'^{y}v^{y})(l_{x+2y-1}-l_{x+2y}). $$

But each of these terms is positive, since dividing each term respectively by $vv'(l_{x+y}-l_{x+y+1})$, $v^2v'^2(l_{x+y+1}-l_{x+y+2})$, ... $v^yv'^y(l_{x+2y-1}-l_{x+2y})$ (all positive quantities) we have quotients

$$v^{y-1}-v'^{y-1},\quad v^{y-2}-v'^{y-2},\quad \ldots.,\quad 1-1.$$

Each, since v is greater than v', a positive term, except the last, which is zero, and therefore the whole expression is positive, that is $(\epsilon)-(\eta)$ is positive, and therefore, looking to interest alone, if a greater rate of interest be realised than provided for in the tables founded on, there will be a greater profit on policies under either of the two schemes this paper treats of, than under the more ordinary method of declaring the bonus in the form of a reversionary sum.

If both the interest and mortality experienced be more favourable than anticipated, there will be under system (ϵ), considering both *original policy and bonuses,* less profit on the mortality, and greater on the interest, than under system (η).

Whether, in that case, (ϵ) or (η) will be more profitable to the Assurance Company, can of course only be determined in each particular case by applying the values given to formula (8).

We have considered the question with a view only to the schemes we have been treating of; but, (8), the formula arrived at will serve equally well to determine whether an Endowment, or an assurance, for which equal cash sums have been paid as premiums, is the more profitable to a Company, and will also give the amount of the difference of profit.

It may be well to illustrate this part of the paper by an actual example.

Two persons, each aged, say 30, take out policies of £100 payable at death in an Assurance Company. Each is charged the same annual premium; and after 5 years, the same cash bonus of, say ·651, is declared on each policy, the one sinks his bonus in the purchase of an Endowment payable at, say age 75, the other buys a reversionary sum (£1. 10*s.*) with his. The office sells both endowment and reversion at Carlisle 3 per cent. The realised rates are $\frac{9}{10}$ Carlisle 4 per cent. On which policy is there greater profit at, say, age 40, and what is the then difference between the two?

The formula for $(\epsilon\sim\eta)$ is here

$$1\cdot5\left(A'_{\overline{35}\,\overline{5}|}\frac{D'_{35}}{D'_{40}}\sim A_{\overline{35}\,\overline{5}|}\frac{D_{35}}{D_{40}}\right)$$

where the first quantity within the bracket (ϵ) is calculated at $\frac{9}{10}$ Carlisle 4 per cent, and the second (η) at Carlisle 3 per cent. When we work this out, we shall find that (η) exceeds (ϵ) in the given time by ·00768, and that, thus, the greater profit on interest realised by (ϵ), is more than counterbalanced by the loss on the sale of the endowments, through the lighter mortality.

As an illustration of our analysis concerning interest and mortality, considered separately, let us take this question; and assuming, first, that the interest anticipated and realised is the same (3 per cent), but that the mortality experience turns out in conformity with the $\frac{9}{10}$ Carlisle instead of the Carlisle table, as founded on: and, secondly, that the Carlisle table represents the mortality, both actual and anticipated, but the interest made be 4 per cent, instead of 3 per cent as founded on—we shall find that, in the first case, (η) will exceed (ϵ) by ·00927, and in the second, (ϵ) will exceed (η) by ·00177.

We shall very briefly glance at another interesting question.

In examining the actual mortality that occurred among policyholders under scheme (ϵ) or scheme (η), we have considered that, $(x+t)$ being any age, $\frac{l'_{x+t}}{l'_x}$ will, by the experience table, correctly represent *under either system* the probability of a life aged x living to age $(x+t.)$ No doubt this would be correct, were there no disturbing circumstances; but this is not always the case. Several Offices give their assurers the option of choosing between scheme (ϵ) and scheme (η), at the time the first bonus is declared on the policy; and some, if we correctly understand the wording of their prospectuses, even permit this to be done at each declaration of profit. In such cases as these, it is evident that though $\frac{l'_{x+t}}{l'_x}$ may represent the probability, taken from the experience of all the lives together, of a life aged x living t years; yet it will be a very different thing when the sickly man is permitted with his cash bonus to buy a reversion, and the robust man an endowment. If it be only done once, and a person having chosen his system be confined to it, the effect of the selection of schemes by the assurer may perhaps die out before long; but it is quite another matter when this disturbing cause becomes perennial by the policyholder making his election at each declaration of profit. With all deference to the opinion of the actuarial advisers of those Offices that allow this latter thing to be done, we cannot but think it an unfortunate permission.

In the first place let us see what is actually allowed. If the bonus be declared, as it usually is, in the form of a reversionary sum; since, by the operation of applying bonus to limiting the term of the assurance, there will be an unusually large number of Endowment Assurances on the books of the Company, we shall have to choose at each declaration between increasing the benefit to the assurers who have at any previous time adopted this system, at the expense of those who have confined themselves to system (η); or by declaring a less reversionary bonus in the one case than the other, disgust those policyholders who no longer wish to pursue system (ϵ). This, we admit, is more a popular than a theoretical objection; but still we think it is none the less forcible, and it is better to avoid the possibility of its being raised.

Again, what are we to understand by the statement that a person may have the choice of either method at each time of division of profit? Does it mean that at each investigation period a person has the power of dealing only with the reversionary bonus then declared, or with all of the others as well? Very possibly these Offices might answer, he can only apply the bonus then declared, to hasten the term of payment of the sum assured, or the extinction of his premiums; but this (though not so bad as allowing him to deal with all bonuses) involves the giving the persons who have already adopted system (ϵ), an advantage over those who, having previously adopted (η), now wish to adopt (ϵ); as, in the former case, from the nature of the formula, all the previous bonuses are dealt with, as well as the one then declared—being surrendered at full value, and sunk in the purchase of a new endowment of shorter term; a thing that a Company would hardly allow to a policyholder, who came and frankly asked for the surrender value of his endowment policy, because, owing to failing health, he could hardly expect to live till it matured, and would therefore prefer to buy, with the sum paid him by the Office, a reversion, or at least an Endowment, payable at a time he would have more chance of reaching.

No doubt, at present, under system (η), an assurer, whatever be his state of health, is allowed to surrender his reversionary bonuses for cash; or, *vice versâ*, with his cash profit, purchase a reversion; but this, an evil in itself, though it has in certain cases its advantages, is much aggravated, when this cash sum is allowed to fructify as an endowment, which at periodical intervals the policyholder has the power of reconsidering, with a view, if need be, to altering it to what may best suit his interests.

In fact, giving this option to an assurer, is practically the same thing as if an Office were to allow its policyholders to surrender their policies at the full amount reserved against them, a thing which, so far as I am aware, no Company would dream of doing; and even should, as in the case of policy surrenders, a deduction be made from the full value, this deduction will be, as in that case, purely empirical.

If, then, either of the schemes that are the subject of this paper, and more especially the first of the two, be chosen, it will be much better to confine a person who has once adopted it, to it, and at each investigation only permit him to apply his bonus to shorten the term of the Endowment.

A comparison arises at once in the mind of everyone between policies on either of these systems, and those corresponding, in which n is fixed at once. Which has the best of it?

Should both assurers buy at cost price, whether that cost price be different from the rates founded on, or not, the time of payment in each case will be the same;* but as the first has to buy at

* A proof of this is hardly necessary; perhaps, however, it may be well to furnish one.

Let two assurers with any Office each pay the premium P_x—the one, purchasing an Endowment Assurance payable at $(x+\nu)$, without additions—the other, an ordinary assurance payable at death, with additions, the amount of this assurance being assumed the same. The latter invests these additions in endowments which became finally payable at age $(x+n)$. Which is least, n or ν?

The premium P_x each pays, is made up in the following way. There is a premium ϖ_x, which being invested at realised rates, will secure the sum contracted for, *at death*. This, together with the addition for expenses, and the margin from which springs the profit, makes up P_x. This margin we shall call f.

In the Endowment Assurance *uncertain*, (we shall take for illustration the *first* of the two systems merely), we have seen the formula is, when all the rates are the same, *i.e.* the realized rates,

$$N_{x+n-1}=(1\times a_x)({}^1\mathrm{b}D_{x+y}+{}^2\mathrm{b}D_{x+2y}+\ldots{}^{\theta}\mathrm{b}D_{x+\theta y}).$$

It must be remembered, as we there said, that these formulæ do not allow the policyholder any profit for the period from $(x+\theta y)$ to $(x+n)$. Let ${}^i\mathrm{b}$ be this profit, and as we suppose the policyholder to reap now all benefit possible, we must add ${}^i\mathrm{b}$ to our formula, which will thus become

$$N_{x+n-1}=(1+a_x)({}^1\mathrm{b}D_{x+y}+{}^2\mathrm{b}D_{x+2y}+\ldots{}^{\theta}\mathrm{b}D_{x+\theta y}+{}^i\mathrm{b}D_{x+n}).$$

Now as f is the only source of profit, we can calculate the value of b. Thus

$${}^1\mathrm{b}=f\frac{N_{x-1}-N_{x+y-1}}{D_{x+y}},\quad {}^2\mathrm{b}=\frac{N_{x+y-1}-N_{x+2y-1}}{D_{x+2y}},\quad {}^i\mathrm{b}=\frac{N_{x+\theta y-1}-N_{x+n-1}}{D_{x+n}};$$

and substituting these values of b in the formula, we have

$$N_{x+n-1}=f(N_{x-1}-N_{x+n-1})(1+a_x).$$

Again, in the Endowment assurance certain, since $(x+\nu)$ is the age at which the sum assured is payable, then, at that age, for ϖ_x (the premium actually necessary to provide assurance at death) the policyholder will have enjoyed a temporary assurance of £1, and an endowment, then payable, of $1-\dfrac{1+a_{x+\nu}}{1+a_x}$ (that is, the surrender value of a whole life

something over prime cost, and receives no bonus, while the second has *his* overpayment returned in this form to him, *theoretically*, n in the latter case should be of earlier date. *Practically*, though we are aware that it involves the somewhat paradoxical statement that a "with-profit" policyholder reaps little or no profit, we do not think there will be any great difference. This can of course only be tested by actual example. Thus, the with-profit premium at age 25 for assurance of £100 at death, we shall assume to be 2·1286, (Carlisle 3 per cent with 25 per cent added), and the bonus declared to be equal to a reversionary sum of £1. 10*s*. per cent per annum. This is, we think, at the very least, not under the average profit that Offices charging such a premium would declare. (The annual premium for such a bonus at Carlisle 4 per cent is ·59154.)

We shall assume the first scheme to be adopted; and this bonus, declared quinquennially, to be surrendered at Carlisle 4 per cent, and the Endowments purchased, and the policy valued, at Carlisle 3 per cent. The following will then show the effect of this system:—

Ages marking beginning of different statuses of Policy.		Time when Policy will be payable.	
25		At death	
30	————	75	At 1st investigation
35	————	71	„ 2nd „
40	————	68	„ 3rd „
45	————	66	„ 4th „
50	————	64	„ 5th „
55	————	63	„ 6th „
60	————	62	„ 7th „

So that the assurance becomes payable at age 62 finally.

We have before us the prospectus of at least one Company of the highest standing, that would grant to a person of the age of

assurance); but he then receives £1, so (as ϖ_x+f is the premium, less expenses, he actually pays), f must be the premium for an endowment of $1-\left(1-\frac{1+a_{x+\nu}}{1+a_x}\right)$. Whence $f=\frac{N_{x+\nu-1}}{(N_{x-1}-N_{x+\nu-1})}\frac{D_x}{N_{x-1}}$. Substituting this for f, in the expression given above, we have,

$$N_{x+n-1}=\frac{N_{x+\nu-1}(N_{x-1}-N_{x+n-1})}{N_{x-1}-N_{x+\nu-1}},$$

$$\therefore\ \frac{N_{x+n-1}}{N_{x-1}-N_{x+n-1}}=\frac{N_{x+\nu-1}}{N_{x-1}-N_{x+\nu-1}}$$

Whence, from the construction of Commutation Tables, N_{x+n-1} must be equal to $N_{x+\nu-1}$, and therefore $n=\nu$.—Q.E.D.

25, for such a premium as the one given above, an endowment assurance payable at 64; and those of others whose rates at 65 closely approximate to this premium, so that the benefit obtainable by an assurer on this principle, should even these highly favourable rates of bonus, surrender values of policy, &c., be allowed in the calculation, does not exceed two or three years.

In the example given above, though we might have at once proceeded to the final age at which the policy will become payable, without finding the intermediate ages, yet we have preferred to give the intervening ages also, as the series will elucidate a peculiarity of these systems of applying bonus, that well deserves to be pointed out. It will be noticed that the same reversionary bonus which, towards the beginning of the series, will bring the term of payment nearer by four years, will, at the end, only bring it nearer by one year. Had we chosen a really equal bonus at each division of surplus, instead of an increasing one, as a constant reversionary bonus necessarily is, this peculiarity would have been even more striking. Of course the reason is to be found in the great cheapness at which endowments for long terms can be purchased, and their much heavier cost when the probability of their being enjoyed is great, as it necessarily becomes when the period of their maturity is close at hand.

We have now, we think, gone over most of the important features that these systems of dividing profit present. They are methods which we consider are likely to grow in favour with the public; and whatever opinion actuaries may hold as to the adviseableness of their introduction, it will be generally admitted that they are extremely interesting methods of dividing the profits realised by an Assurance Company.

On the Distribution of Profits in Mutual Insurance Societies. By Professor Pell, *of the Sydney University.**

[Read at the meeting of the Philosophical Society, on Wednesday, 7th December, and reprinted from the *Sydney Morning Herald*, 10th December, 1864.]

THERE is no part of the subject of Life Insurance which has occasioned so much difficulty, and given rise to so much diversity of opinion and of practice, as that of the distribution of profits.

* We reprint with pleasure this paper, for which we are indebted to the kindness of a correspondent. The arguments contained in it appear to us very ingenious and forcibly put, although far from convincing. It would require more time than is at present at our command to give a full examination of Professor Pell's arguments, but we have briefly indicated the points in which they appear to be defective.—Ed. *J. I. A.*

The methods which have been adopted are very various, and very few of them seem founded upon any intelligible principle.

Many attempts have been made of late years to form an exact theory on this subject, and to deduce systematic rules; but it cannot, I think, be yet said that there is any method which is generally recognised as theoretically correct, and capable of application in all cases.

In order to form correct rules for the distribution of profits, it is necessary, in the first place, to lay down some fundamental principle, depending upon the nature of the contract of insurance, upon which the method of distribution must be based, and in the next, to express the principle by means of formulæ capable of practical application. Writers upon this subject have generally assumed, without comment or controversy, that the fundamental principle is the following. If, at any time, upon investigating the affairs of a Mutual Insurance Society, a surplus is found to exist, there should be returned to each member of the society that portion of the surplus which he contributed. A sum being reserved sufficient to cover all the liabilities, the surplus, if any, is considered not as the property of the society, and not strictly speaking as profit, but as the property of certain members held in trust, to be returned to them according to the proportions in which they contributed to it. If it turn out at any investigation that the members have been paying too much in the form of premium, or that the sum reserved at the preceding investigation was more than the event has shown to have been necessary, then the surplus is to be regarded as a separate fund from which each member is to be repaid in proportion to his contribution. When the amount of the surplus has been ascertained, nothing remains then but to determine how much each member contributed towards it. But, with the greatest respect for those eminent writers who have made this principle the foundation of their investigations, there are many considerations which have driven me to the conclusion that it is in reality fallacious, and in many cases is, and necessarily must be, ignored. I shall endeavour to show that where there is no antecedent agreement as to the mode of distribution of profits, the adoption of the principle against which I am contending involves a violation of some of the rules which are universally recognised and adopted in settling mutual contracts, and in regulating the affairs of monetary and commercial associations. I cannot see that there is anything so entirely peculiar in the mutual contract of life assurance to take it out of all ordinary rules, and to require or justify suc-

cessive revisions and amendments according to circumstances as they arise.

In stating my objections to this principle, I shall endeavour to show (1) that it cannot possibly be fully carried carried out; (2) that if in regulating the affairs of any society it could be pushed to all its legitimate conclusions, it would be found that the society was not in reality an insurance but an investment society; (3) that the principle must be fallacious, for if consistently applied, it would, under some circumstances, lead to results, the justice, or even the legality, of which could not be maintained.

In a paper by Mr. Sheppard Homans, actuary of the Mutual Life Insurance Company of New York, published in the London *Assurance Magazine* in October, 1863, a most elaborate method is given of estimating the contributions of the several members to the surplus fund. "It appears," he says, "that the contributions or over-payments of policies during a bonus period may in general be found thus:—*Credit* each policy-holder (1st) with the amount actually reserved at the last preceding distribution of surplus as the then present value or re-insurance of the policy; and (2nd) with the *effective* (or *full*) premiums paid since that time, both sums being accumulated at the actual current rate of interest, to the date of the present distribution; and charge him (1st) with the actual cost of the risk to which the company has been exposed, during the interval, determined by means of a table representing the rates of mortality and interest actually experienced; and then (2nd) with the amount now reserved as the present value of the policy. The difference between the sum of his credits and the sum of his debits determines the over-payment or contribution from the policy proper."

This is perhaps the clearest statement that has been made upon this subject, and amounts to this. The premiums were originally settled according to a certain rate of interest, and a certain table of mortality, with a margin or loading added to cover expenses and contingencies. At the former investigation a sum equal to the liability upon each policy was calculated upon the same basis and reserved to the credit of the policy. At the present investigation it is found that during the interval the rates of interest and of mortality have been more favourable than those assumed, and a new scale of premiums is calculated upon the experienced rates, being the scale according to which the members should have paid if those rates could have been foreseen. To each member there is returned the difference between what he has paid and what it is

supposed that he ought to have paid, together with the superfluous accumulations of interest upon the sum reserved to his credit at the former investigation. But even this elaborate method is very far from attaining to that mathematical equity which is intended, for it must be remembered that the sums to the credit of the policies at the former investigation would have differed, not only in amount, but in proportion, if the events of the succeeding bonus period could have been foreseen; and, therefore, to carry out the principle fully, those sums should be revised, which would greatly alter the results, and would lead to further, and practically interminable, complications.

[For the reasons stated in our last number we agree with Professor Pell that Mr. Homans's plan falls short of strict equity; but we do not think it is fairly open to the objection here made against it.]

By the effective premiums Mr. Homans explains that he means the actual premiums with a certain per centage upon the amount insured deducted to cover expenses. This mode of apportioning the expenses is correct as far as it goes—there is no reason why a policy upon an older life should be charged more for expenses of management than one upon a younger, the amounts assured being the same. But it falls far short of settling the question of the apportionment of expenses so fully, as the strict mathematical equity which the system proposes to carry out seems to require. It would probably be found, upon a careful examination, that the actual expense to the society occasioned by any policy has little, if any, dependence upon the amount assured; but that the expense is pretty nearly the same upon one policy as upon another. Excepting, perhaps, in very large societies, where the expenses are relatively light, it would in many cases be found that if the policies for small amounts were charged with the expenses which they had really occasioned, there would be nothing of profit left to them, although a considerable general surplus might exist. It might and probably would appear in some cases, that such policies had been an actual loss to the society. In seeking to determine the real contributions to surplus by the several members, there is no reason why these considerations should be neglected, and their omission seems to render the results obtained of no practical value.

[There is much truth in these remarks; but they are probably exaggerated. If they were strictly correct, it would follow that an Insurance Company ought not to issue any policy, or, at all events, ought to allow no profits to any policy below a certain amount.]

To carry out the method explained by Mr. Homans, it would be necessary at an investigation to form a new table of mortality, upon the actual experience of the society since the former investigation. It is difficult to see how, in so short a time and with so limited a number of lives, any table could be framed of any value whatever; but without such a table it would be impossible to apportion the actual losses or cost of insurance during the interval, in accordance with the assumed principle. The losses might be apportioned in the same proportion as if the assumed rates of mortality had been actually experienced; and this would seem the only practicable method in any case where the number of members is too small to form the basis of a new mortality table; but it is to some extent arbitrary, and very imperfectly in accordance with the principle upon which the system rests.

[No doubt much difference of opinion exists as to the number of lives sufficient to form a basis for a mortality table; but for the purpose Mr. Homans has in view, probably the experience of almost any Company during a bonus period of, say, five years, would suffice.]

There is another consideration which cannot be consistently neglected, if a really accurate estimate is to be formed of the contributions of the several members of a society to the surplus fund. It is well known that it is much easier to find safe and good investments for smaller sums than for larger ones, and that as the funds of the society increase, there is an increasing difficulty in finding safe and profitable investments. It may happen, and indeed has happened, that a portion of the contributions of the earlier members of a society are more profitably invested than the funds received from those who come after them. If, then, a member is considered to retain any special property in the money which he has paid to the society, the earlier members might well contend that they are entitled to the full benefit of the higher rate of interest at which their funds were invested, and it might even perhaps be allowed that as the older investments expire, the earlier members are entitled to have their funds reinvested in the most profitable securities which the society can find. The justice of this claim can hardly, I think, be disputed if the principle against which I am contending be allowed. The newer members, whose funds are necessarily invested at a lower rate of interest, can hardly be said to have contributed to any surplus which may exist, according to any higher rate of interest, than what those funds have actually yielded. In England, where the rate of interest upon money invested in good securities is tolerably

uniform, these considerations would not perhaps be of much importance, but the case is very different in this colony.

[If the difference as to rate of interest on investments exists to any marked extent, nothing would be easier than to allow the old members the benefit of it. But, certainly, as regards their current payments they ought to be on an equality with the new members. It seems also too much to require that as the old investments at high rates of interest are repaid, the best of the new investments should be appropriated to the old members. In England the state of things supposed does not exist.]

To take account of the various rates of interest in calculating the contributions to surplus, would involve difficulties, which would probably be found to be insuperable, but their neglect would render the results of no practical value.

If the principle is once admitted that the original contract of assurance is to be varied, to the extent that every member is entitled to claim as his own, anything which he has paid in excess of what experience has proved to have been requisite, we shall be driven to conclusions totally inconsistent with the nature of insurance. If there has been a proportionally lower rate of mortality among the younger lives than amongst the older, then the younger lives should be charged with the losses upon their own class only. The member who insured at the age of thirty should be charged with no losses except those upon policies taken out at that age. To push the principle a little further, those who insured at the age of thirty, five years ago, should not be charged with losses upon policies dated ten years back; and so on by successive subdivisions, until we should come at last to the conclusion that no member can be equitably charged with any losses, except those upon his own policy; which would amount to this, that upon the death of a member there should be paid back to his representatives the exact sums which he paid in with interest added and expenses deducted.

Suppose, again, that the members of the society live in two towns, and that experience proves that the rates of mortality are more favourable in one town than in the other. If any account is to be taken of the proportion in which the members have contributed to surplus during any period, it cannot be denied that those who live in the healthier town should be charged with losses according to their own more favourable rates of mortality. On the same principle, the towns should be divided into sections, the members residing in which should be charged with their own losses only. For similar reasons we should go on to subdivide into

streets, and then into houses, and finally into single policies, arriving thus at the same conclusion as before; which is, indeed, the final conclusion to which the assumed principle must necessarily lead, and the only perfectly equitable solution of the whole difficulty.

[There is certainly no ground for speaking of the original contract of assurance being varied, whatever method of dividing the surplus be adopted. With regard to the considerations as to the exact cost of insurance to be charged to the insured, it must be left to the actuary to decide how far what is mathematically just is practically possible. The subdivision of the assured into small classes may be carried, if thought just or desirable, to a considerable extent, but must, of course, from the nature of the case cease long before the classes are reduced to individuals.]

I shall now endeavour to show that the fundamental principle which has been so commonly assumed, that any surplus which may accumulate during any period is not really the property of the society, but is merely held in trust to be returned to the members, is inconsistent with the nature of the mutual contract of insurance usually entered upon; and not only is not, but in many cases could not reasonably or even legally, be adhered to.

If during any period a surplus is accumulated on account of higher rates of interest or low rates of mortality, or excess of loading, then according to the assumed principle, this is the property of some or all of the members in a certain proportion; and this proportion, it must be observed, is quite accidental, depending upon a variety of circumstances, which cannot in any case be foreseen. Now, suppose that upon the eve of the day upon which the investigation is to take place, a loss exactly equal in amount to this surplus which has accrued, is incurred through some accidental cause, as the failure of a bank. There could be no possible doubt, I conceive, that the surplus should be appropriated to make good this loss. To the justice and legality of such an appropriation, no objection could possibly be made. But according to the assumed principle, such an appropriation would be grossly unjust, being an application of the private property of certain members, to make good a partnership loss. If there had been no surplus, then all the members must have borne the loss, in a certain proportion. The fact that there happens to be in the hands of the society a fund, the private property of the members, cannot alter the proportion in which the loss should be borne, and to apply this fund to cover the loss would be clearly unjust, unless the proportion were the same, which could only happen by the merest accident. According to the principle, the fallacy of which I am

endeavouring to prove, the only proper course would be, to apportion the loss, and to apportion the surplus separately, and so to balance the accounts. This would lead us to the absurd conclusion, that some of the members would have to bear a certain loss, or abate somewhat of their claims according to the original contract, although the society on the whole was in a perfectly solvent state, whilst to apply the surplus to cover the loss, would be at once to affirm that the surplus fund is, and that it is not the common property of the society—common property to pay losses, but not common property to be distributed as profit.

Again it might be found upon investigation, that by reason of high rates of interest, a certain profit had accrued, but that in consequence of a high rate of mortality, a loss had been incurred. To follow the obvious and necessary course of applying the profit to cover the loss would involve similar contradictions.

[This is very ingenious; but it entirely leaves out of sight the fact that the assured are charged in the first instance a certain amount, viz.: the loading of their premiums, on purpose to provide for contingencies, including such as the failure of a bank; and if the surplus arising from the overpayments of the assured is applied to make good a deficiency of the kind supposed, it is only fulfilling the very object for which it was intended. Of course there is no divisible surplus until all the losses from adverse contingencies of every kind have been paid or written off. The assured thus virtually share the losses in the same proportion as they would share the profits.]

The word *equitable* has been much used by writers upon this subject, and it seems always to have been assumed that equity consists in returning to every member so much as he may have paid in excess of what has in fact proved to be sufficient; or, in other words, to reform the original contract by the light of new experience. I think that I have sufficiently shown that this principle of equity, if carried out to its full extent, and I cannot see that there is any particular point at which we can stop short in its application, is inconsistent with the existence of any contract of insurance at all. My idea of equity, as applied to this subject, is that the original contract should be rigidly adhered to, without reference to subsequent events. A member contracts to pay an annual premium calculated upon the basis of certain rates of interest and mortality. If on the whole the rates of interest and mortality prove favourable, a surplus will accumulate, and will be the common property of the society, independently of the sources from which it may have been derived, and should be dealt with in the same manner as if it had been caused by a rise in the value of securities, or otherwise accidentally.

[As we have already stated, we think the question does not arise of varying or reforming the original contract, which is simply to pay the sum assured; and all will agree that this original contract ought to be rigidly adhered to. The last sentence is a mere begging of the question.]

The observed rates of interest and mortality, and calculations respecting the expense of management, may afford very useful data for the formation of new societies, or for determining the conditions upon which new members should be admitted, but I cannot see that they have any relevancy whatever in estimating the proportion in which an existing surplus is to be divided.

I will endeavour by an illustration to make my meaning more clear. Suppose that a number of persons should combine together to form a society for the purpose of cultivating land, so as to provide themselves with such quantities of wheat, maize, and hay as they might require, each member agreeing to take so much of each commodity at a certain fixed price, to be paid annually in advance—the agreement to continue in force during a certain time, and the profits, if any, to be divided equitably at fixed intervals.

If the profits were to be divided according to the principle attempted to be applied to life insurance societies, it would be necessary when a surplus had accumulated to make an accurate estimate of the cost of producing each commodity, and to charge each member accordingly, so as to ascertain how much each had contributed to the surplus. If wheat were found to have cost much less than had been anticipated, then the member who had agreed to take wheat only, would receive a large share of the profit. If the cost of hay agreed exactly with the estimated price, then the man who took hay only would receive no profit. If there were a loss upon the maize, then those who took maize should, on the same principle, instead of sharing in the profit, be required to make good the loss. It would be contrary to the principle to appropriate the contributions from the wheat consumers to pay the losses occasioned by the maize consumers, for every man's contribution should be regarded as his own. I think it will be conceded that this mode of apportioning the profits would be directly at variance with our most commonly received notions of equity, and that the principle involved would, if fully carried out, render mutual contracts to be performed in future practically of no effect.

An estimate of the actual cost of the several commodities produced would furnish useful data in forming a new society for similar purposes; and the calculations of how much each member contributed to surplus might afford an interesting arithmetical

exercise; but in determining how the profit in hand should be divided such estimates and calculations would be wholly irrelevant.

[We cannot agree with the author's view of his illustration; but think, on the contrary, that in the absence of any agreement except that the profits should be *equitably divided,* all the profits should clearly go to the purchasers of wheat. Whether the purchasers of maize should be surcharged would depend on the intention of the partners to be deduced from the terms of the contract.]

I will now endeavour to explain what I consider to be the true principle upon which profits should be distributed in Mutual Life Insurance Societies.

Suppose that a number of persons mutually insure their lives. Each member contracts to pay a fixed sum annually during his lifetime, and the society contracts to pay a certain sum upon his death. At the end of five years, suppose a fund will have accumulated in the hands of the society, and at the same time each policy will have acquired a certain value. Suppose now that it were determined to wind up the society, and to divide the fund. Each member should in the first place of course receive the present value of his policy, or the value of his claim against the society. If the payment of these sums should exhaust the fund, there would be an end of the matter; the society would have proved exactly solvent and no more, and no one would have lost or gained anything by it.

The present value of the policy on the life of any member is his share of the fund, being precisely what he would lose if the fund were annihilated, and what he would receive if the society were wound up. It is, in fact, the sum which he has invested and risked in the concern. It is not, of course, so great as the full amount of the premiums which he has paid with interest added, for a portion has gone to pay for losses by deaths during the five years, and for this he has received an equivalent, viz., the security which he has enjoyed during the five years, that in case of his death a certain sum would be paid to those for whom it was his intention to provide.

[Surely the premiums paid must be considered the "amount invested"; and although the insured has received an equivalent for a certain part of his payments in the guarantee afforded by the Office, how can it be ascertained, except by some such process as Mr. Homans's, that the proper sum has been charged the assured in estimating the value of his policy?]

It is not my object in this paper to consider how the present value of a policy should be determined, but in a merely business-like point of view, it should be such a sum as would enable a

member in case his own society was wound up, to compound with another similar society to insure his life at the same premium which he had hitherto paid, although his age was greater by five years.

If upon winding up a society it were found that the funds were not sufficient to pay back to each member the present value of his policy, it is quite clear that each must abate proportionally. Whatever per centage is wanting on the whole, the same per centage must be deducted from each. I cannot see that there would be any other just or legal way of apportioning the loss, and it appears to me equally clear that if instead of a deficiency there were a surplus, it should be distributed in the same proportion; that is, in proportion to the present value of the policies or to the sums invested and risked in the society by the several members. This is precisely in accordance with universal practice in all cases where several persons are jointly interested in any undertaking. Profits are always divided in the same proportion in which losses, if incurred, would be apportioned. It may be urged, as an objection to this method, that, under certain circumstances, some of the members would receive a larger share of the surplus than they had contributed; but it would always be found in such cases that if a loss had been incurred, the same members would necessarily have borne the larger share of the loss, and in the same proportion.

[It appears to us that a "winding up" is so foreign to the idea of a Life Assurance Company that any considerations drawn from it cannot fail to mislead. Suppose, on the contrary, that the Office is to be carried on; that it is solvent, but has no surplus; that in this state of things a year's premiums are received on one day, and lost the next day through the failure of a bank or similar cause: how ought the deficiency to be made good? Clearly, by calling on each member to pay the lost premium again, which would be a very different thing from making a levy in proportion to the values of the policies. This argument, as far as it goes, would be in favour of dividing the profits in proportion to the premiums paid; but according to our view, whether a loss has to be assessed, or a profit to be distributed, regard should be had to the manner in which the loss or the profit has arisen.]

In a purely proprietary company, under proper management, the insured incur no appreciable risk, and receive no share of the profit. The shareholders take all the surplus to which they have contributed nothing. They are simply paid for the risk which they have incurred, small though it be. The insured enjoys a security for which he pays, and the shareholder undertakes a certain risk, for which he is paid. To say that this is inequitable, because the

shareholder had contributed nothing to the surplus, would not be more absurd than it would be to condemn as inequitable a system under which a member of a Mutual Life Insurance Society might, under certain circumstances, receive a larger share of the profits than he had himself contributed. When a man becomes a member of a Mutual Insurance Society he not only insures his life, but, to a certain extent, he engages in the business of life insurance; he is at once insurer and insured, and as insurer he may be considered at any time to have invested in the business a sum equal to the present value of his policy, and is entitled to a proportional share of the profits in the same way and for the same reason that a shareholder in a proprietary company is so entitled.

[As we have stated above, we cannot admit that the value of the policy is the sum invested by the mutual assurer. The whole argument appears to be vitiated by the neglect to notice the peculiar conditions under which the business of life insurance is transacted by the mutual assurer. It must never be forgotten that confessedly excessive premiums are charged in the outset—so calculated as to reduce to a minimum the possibility of any deficiency, inasmuch as they contain, not only the charges for the bare cost of insurance, and for the expenses of management, but also a further sum for contingencies, which, if not absorbed by those contingencies, is available for return to the members.]

This present value is small at first, and increases with the duration of the policy, but where the premium is paid in a single sum, the policy has at once a considerable value. In estimating the present values of the policies at any investigation it is necessary of course to take into account any previous bonus additions which may have been made, every such addition amounting in reality to a new paid-up policy.

Any system of distribution which depends upon the assumption that the loading is the principal source of profit, the "bonus producing power" of the policy, as it has been called, is quite inapplicable in this colony. In England the rates of interest upon good securities are tolerably uniform, and the probability of any great fluctuation very remote, so that it is safe to calculate upon a rate of interest a very little less than what may actually be obtained. But here, although a comparatively high rate of interest may be obtained, its continuance cannot safely be depended upon, so that there is necessarily a wide difference between the assumed and the experienced rates. This has been and will no doubt continue to be, for many years, one of the largest sources of profit to Life Insurance Societies in this colony.

[If the valuations are made at 3 per cent interest, the loading of the premium is certainly not the chief source of profit; and even when 4 per cent is the rate employed, the profits from miscellaneous sources will form a considerable percentage of the total profit. There are many actuaries in the United Kingdom, who consider that an average rate of interest exceeding 4 per cent may safely be expected to be realized for many years to come, but that nevertheless it is desirable to make their valuations at 3 per cent.]

By a method explained by Mr. Meikle in a paper recently published in the *Assurance Magazine,* the contributions from interest alone by the several members of a society may be calculated with exactness upon the supposition that the members die according to the assumed rates of mortality. By a similar method some general results may be obtained as to the effects of more favourable rates, and of loading. I cannot within the limits of this paper enter fully into this part of the subject, but can only state some results in a very general form. The contributions to surplus from a very high rate of interest are larger in proportion to present values from older and from paid-up policies, than from ordinary policies of short duration. The reverse is the case with respect to profits arising from loading and from low rates of mortality; the contributions from these sources from policies of very short duration, being larger in proportion to their present values, than from older or from paid-up policies. The losses which might be occasioned by high rates of mortality would be in the same proportion. There is a greater proportional profit on the newer policies, but, at the same time, there is a greater risk. According to the method of distribution which I propose, the younger policies would under such favourable circumstances receive less, and the older ones more, than they had contributed; and the general effect would be that, until a member attained to the average age, his bonus additions would be less than an equivalent for his contributions to surplus; but afterwards the balance would be in his favour. This would be the case in an old society with policies existing of all ordinary durations.

None of these results respecting contributions, according to my view of the matter, afford any argument for or against the method of distribution which I advocate. I have stated them merely for the purpose of explaining the reason of certain effects which that method produced at the recent investigation of the Australian Mutual Provident Society. The circumstances of that society are wholly exceptional and I believe unprecedented. The rates of mortality during the past five years have been unusually low, and the profit from this source upon policies of short duration, although not very considerable, is large in proportion to their present values.

The method, therefore, of dividing the surplus in proportion to present values has in this case operated favourably to the older and more valuable policies. Under ordinary circumstances, where there is the usual proportion of old and of new policies, this would not have produced any very marked effect. The business of the Australian Mutual Provident Society, however, has increased so enormously during the last few years, and the preponderance of very new policies is so excessive, that a policy of six years' standing may be regarded as an old policy, and one of ten years as very old.

On account of the favourable rates of mortality which the society has experienced, this new business has proved very remunerative, and of the profits from this source, the policies of longer duration have in some cases received a larger share than what they actually contributed. Those who think there is anything really inequitable in this, should remember that those who have received the larger share of the profits have also incurred the larger share of the risk. The accumulated capital, the property of the earlier members of the society, was the guarantee fund, without the security of which the greater number of the newer members would never have insured their lives in the society at all; and upon this fund, if any loss had been incurred through a high rate of mortality, the larger share of the loss must have fallen. Any such loss must have been made good, if possible, out of the surplus arising from high rates of interest, to which the newer members have contributed comparatively little. The larger share of such losses would unavoidably have fallen upon the earlier members, and, therefore, they are entitled to participate in profits in proportion to the sums which they have invested in the society and exposed to such risks. It would be very unjust that the newer members should share fully in the profits arising from favourable rates of mortality, whilst they are to a great extent secured against losses, by the funds already accumulated.

[It must not be forgotten on the other hand that unless the old members had considered it to their own advantage that new members should be admitted, they would have excluded them. Having thus for their own benefit admitted new members, they surely ought not to refuse to allow these to participate fairly in the profits to which they have contributed.]

Even supposing that the real contributions to surplus from every policy could be estimated, it would be inequitable to divide profits in that proportion, unless there was an understanding that losses—in any way incurred—should be separately estimated and similarly apportioned. If the surplus from one source, or from all sources,

is merely held by the society in trust to be returned to the members in certain proportions, then clearly if a loss be incurred through rates of mortality in excess of the assumed rates, or otherwise, this loss must be the private debt of the members, to be paid by them in certain proportions, independently of any surplus which may have arisen from other sources. Such a system would be inconsistent with what seems one of the fundamental principles of insurance, viz., that all losses shall be borne in common and paid for out of the common fund; and could not reasonably or even legally be carried out except under a distinct antecedent agreement.

[If the considerations we have pointed out above be borne in mind, it will be seen that the usual way of estimating the divisible profits (whatever may be the mode of distribution) does virtually apportion any losses that have happened in the same proportion as the profits are distributed; the only restriction being that, the premiums being heavily loaded, it is assumed that no losses will occur in any bonus period so large as more than to absorb all the profits.]

This appears a convenient opportunity for stating briefly the method of distribution of profits which appears to ourselves to combine in the highest degree the requisites of justice and facility of application, and which we have found in practice to give very satisfactory results.

If the average rate of interest at which the total funds, inclusive of bankers' and agents' balances and other unproductive assets, are improved, exceeds the rate at which the valuations are made, it is clear that a profit will be realized in the nature of excess of interest on the amount of the funds at the last valuation, to which profit the new members have contributed nothing. Let the amount of the profit so earned by each of the old assured still remaining on the books be ascertained, and appropriated to his policy. For example, if the valuations are made at 3 per cent interest, and the average rate realized has been £4. 8*s*. per cent, then the reserve made for each participating policy at the last valuation (five years ago) is to be multiplied by $\cdot 0809 (= (1\cdot 044)^5 - (1\cdot 03)^5)$; and the product will be the profit to be in the first instance appropriated to the policy. The sum of all these amounts being found and subtracted from the surplus divisible among the assured, there will remain a sum which may be fairly divided among all the assured in proportion to the premiums (without interest) they have respectively paid since the last valuation. As regards persons of the same age at entry, it is clear that, apart from the effect of selection, the profit on their current premiums must be nearly the same, whenever their policies were effected; and if the premiums are loaded with a percentage on the net premium, or approximately so, the distribution of the surplus in proportion to the premiums paid, will give very fair results. This method will have the effect of giving larger cash bonuses to the policies the longer they have been in force; but not unreasonably or unfairly so.

We notice that Professor Pell does not state how he would divide the profits at successive valuations, whether on each occasion in proportion to the value of the policy, or whether at subsequent valuations, he would

distribute the profits in proportion to the increase in the value of the policy since the date of the last valuation. His argument would point to the former of these methods; but we believe that if that method were applied to an old Office, its injustice would be so obvious that its most strenuous defenders would abandon it.—ED. *J. I. A.*

Further Considerations on M. Violeine's Solution of a Problem on the Rate of Interest in Loans repayable by Instalments. By PETER GRAY, F.R.A.S., *Honorary Member of the Institute of Actuaries.* (See *ante*, pp. 92 to 99.)

IT has been represented to me that, in view of the influential position occupied by M. Violeine, the *quasi* official and authoritative character of his book,* and the currency likely to be thereby imparted to the erroneous principles employed by him in the treatment of problems of the class to which the first of those I have discussed belongs;—I say it has been represented to me that in these circumstances it might be well to be a little more explicit as to his manner of proceeding. In my discussion of the problem in question I did enough, I believe, to shew that M. Violeine's solution is certainly erroneous. I admit however that I did not succeed to my own satisfaction (nor apparently to that of others) in precisely hitting the blot in it, and, as it were, reducing it to its elements. I think that from the consideration I have since given to the matter, I am now able to point out the fallacy by which it is vitiated, and I therefore willingly accede to the suggestion that has been made to me.

The problem may, for our present purpose, be stated as follows:—A person borrows 10,000,000 and grants in return an annuity for forty years, of which the first, second, third, . . . mth payments are 417500, 424680, 431620, ... $410080+7540m-120m^2$, respectively;† and we are required to determine the rate that he pays for the accommodation. Or, to put it more technically, at what rate does the annuity amortize the loan?

The required rate may be described as either the rate at which the sum borrowed must be improved so as just to suffice for the payment of the annuity; or as the rate at which, at any epoch, the values of the annuity and the sum borrowed are equal.‡ So that, if the beginning of the term be chosen as the epoch of reference, the rate will be that at which the *present value* of the annuity is equal to the sum borrowed; and if the end of the term be chosen, the rate will be that at which the *amount* of the annuity in forty years, and that of the sum borrowed, in the same time, are equal.

* M. V. is described, on the title-page of his book, as "*Membre de l'Ordre impérial de la Legion d'Honneur, chef de Bureau au Ministère des Finances, Auteur de plusieurs ouvrages sur les operations industrielles*"; and a work of 232 pp., 8vo., now before me, bears the following title:—"*Crédit Foncier de France.—Problèmes relatifs aux Intérêts Composés, et plus spécialement aux operations du Crédit Foncier.—Nouvel emploi des Tables de Violeine pour la solution des Problèmes.*"

† See pp. 94, 95.

‡ This is in accordance with a general principle which is almost axiomatic; namely, that if two benefits (*certain*, not *contingent*,) be, at a specified rate, of equal value at any one epoch, they will also, *at the same rate*, be of equal value at any other epoch.

It is shown, p. 97, that, referring the values to the former of these epochs—the beginning of the term—at 3·72585 per cent, the present value of the annuity is 9,999,978, as against 10,000,000, the sum borrowed; and if the values be referred to the latter epoch, we shall find that, at the same rate, 3·72585 per cent, the amount of the annuity is 43,199,554, as against 43,199,637, the amount of the sum borrowed in forty years. We conclude therefore that 3·72585 is the rate required, very nearly.

Now the rate assigned by M. Violeine is 3·7613, and we must therefore examine the principle of his solution.

M. Violeine proceeds as follows:—He finds that the amount of the variable annuity at the end of the term, at 3 per cent, is 36,752,392. He then divides this amount by 75·401,260, the amount of a uniform annuity of 1 in forty years, also at 3 per cent. The quotient, 487,424, is obviously the uniform annuity, during the same term, which, at 3 per cent, is equivalent to the given variable annuity. He next finds that at 3·7613 per cent the amounts of the uniform annuity and the sum borrowed, at the end of the term, are equal; in other words, that at this rate the uniform annuity would amortize the loan. And he hence concludes that this is the rate at which the variable annuity amortizes it.

It is here that the fallacy lies. Never surely was inference more unwarranted. It does not in the least follow from the premises. I will put M. Violeine's argument more briefly, thus:—Two benefits (which are not identical), A and B, are equal *at one rate of interest.* One of them, B, *at another rate,* is equal to a third benefit, C. And M. V.'s conclusion is that *therefore, at this rate,* A also is equal to C; from which it would follow of course that *at this rate* A and B are equal.

To make good his inference M. Violeine must be prepared to maintain the following proposition, that if two benefits (which are not identical) are equal at one rate of interest, they are also equal at any or every other rate —a proposition which we know to be absurdly untrue. We therefore altogether reject the rate, 3·7613, assigned by M. Violeine.

I take the present opportunity of giving the formulæ requisite for the general solution of problems of the class to which that under consideration belongs. The enunciation is as follows:—

A loan of S pounds, bearing interest at i per £1, and to be amortized in n years, is represented by k bonds of £a each, ($ak=S$). The bonds are to be paid off, with a premium of p per £1, as follows:—m the first year, $m+d$ the second, $m+2d$ the third, ... $m+(n-1)d$ the nth. What is the cost per cent of the loan to the borrower?

It is necessary to observe that m and d are connected by the equation

$$nm+\frac{n(n-1)}{2}d=k,$$

in which n and k are known. So that, one of those quantities being given or assumed, the corresponding value of the other is easily determined.

The two sets of annual payments—the interest, and the amount of the bonds paid off with the premium upon them—form together a variable annuity, to last n years, the rate of interest involved in which, so that its value at any epoch shall equal that of the sum borrowed at the same epoch, and at the same rate, is the rate required. It is convenient to assume the

beginning of the term as the epoch of reference. The required rate is then that at which the present value of the annuity shall be equal to the sum borrowed.

We thus have to satisfy the equation

$$V=S,$$

where V is the present value of the annuity expressed in terms of i, the required rate per £1.

By the general formula for the value of an annuity whose payments are b_1, b_2, b_3, &c., adapted to this case, we have

$$V=\frac{b_1}{i}+\frac{\Delta^1 b_1}{i^2}+\frac{\Delta^2 b_1}{i^3}-v^n\left(\frac{b_{n+1}}{i}+\frac{\Delta^1 b_{n+1}}{i^2}+\frac{\Delta^2 b_{n+1}}{i^3}\right);$$

and we have to find the value of i which makes this expression equal to S, the sum borrowed.

We easily find for the $(n+1)$th payment:—

$$b_{n+1}=\{S-an[m+\tfrac{1}{2}(n-1)d]\}i+a(m+nd)(1+p);$$

differencing twice with respect to n, we get,

$$\Delta^1 b_{n+1}=a[d(1+p)-(m+nd)i],$$
$$\Delta^2 b_{n+1}=-adi, \text{ constant.}$$

And if we make in these $n=0$, we get,

$$b_1=Si+am(1+p),$$
$$\Delta^1 b_1=a[d(1+p)-mi],$$
$$\Delta^2 b_1=-adi.$$

These expressions may be verified on M. Violeine's problem; but they are equally applicable to any problem of the same kind. By the use of them the preliminary work on p. 94 is saved.

HOME AND FOREIGN INTELLIGENCE.

COMMERCIAL UNION ASSURANCE COMPANY.

Established 1862.

Report to the Directors by the Actuary, March, 1868.

Part I.

Detailed information upon the Business and constitution of the Office.

(1.) *Business and Regulations.*

The first Policy was granted by the Life Branch on the 1st day of May, 1862.

The number and amount of the several kinds of Policies granted by the Company, of those under which its liability has run-off, and of those remaining in force on the 31st December last (which are the subject of the present valuation), are given in the following Tabular statements.

TABLE OF POLICIES GRANTED.

Description.	No.	Amount.
		£
With Participation.		
For the Whole Term of One Life . .	1,769	1,268,000
For various Terms and Contingencies .	89	73,750
Without Participation.		
For the Whole Term of One Life . .	310	345,070
For various Terms and Contingencies .	271	366,390
	2,439	2,053,210

TABLE OF POLICIES VOID.

Description.	No.	Amount.
		£
With Participation.		
For the Whole Term of One Life . .	300	179,560
For various Terms and Contingencies .	11	6,700
Without Participation.		
For the Whole Term of One Life . .	61	88,250
For various Terms and Contingencies .	97	135,792
	469	410,302

TABLE OF POLICIES IN FORCE.
At 31st December, 1867.

Description.	No.	Amount.
		£
With Participation.		
For the Whole Term of One Life . .	1,469	1,088,440
For various Terms and Contingencies .	78	67,050
Without Participation.		
For the Whole Term of One Life . .	249	256,820
For various Terms and Contingencies .	174	230,598
	1,970	1,642,908

(2.) *Funds and Interest.*

The Funds of the Life Branch, after crediting the account with the Premiums which were due but had not been paid owing to the Days of

Grace not having expired, and also with the Interest which had accrued on the investments up to the 31st December, amounted on that day to £173,501. The following Table shows the progress in the accumulations of the

LIFE BRANCH.

Amount of the Funds on the 31st December,					1862,	£7,732
Ditto	„	„	„		1863	24,078
Ditto	„	„	„		1864	52,993
Ditto	„	„	„		1865	88,275
Ditto	„	„	„		1866	124,896
Ditto	„	„	„		1867	169,623

NOTE.—In the last amount, the Premiums due and Interest accrued are not included, as these items are omitted in the sums for the corresponding dates of the previous years.

The rate of Interest realised upon the Life Funds, throughout the whole period, has averaged nearly £5 per cent., and the investments on the 31st December were such as to yield £4 16s. 6d. per cent. As the rate of Interest at which the Premiums are expected to improve has been assumed in the valuation at 3 per cent. only, it will be seen that a very large margin of future profit has been provided in the item of Interest alone, though in future it can scarcely be expected that the present average rate will be maintained.

(3.) *Mortality.*

It has been a fact long known to Actuaries that the mortality of recently-selected lives falls considerably below that of any tabular rate. The Life Premiums of the Commercial Union are based upon the Carlisle Table of Mortality, and the results experienced by the Office, as compared with the mortality expressed by that Table, are as follows:—

TABLE.

Year.	Average Amount at Risk.	Expected Claims by the Carlisle Table.	Actual Claims.
1862	69,900	873	Nil.
1863	323,280	4,024	2,700
1864	640,942	8,227	1,300
1865	926,396	12,157	1,200
1866	1,119,179	14,940	3,600
1867	1,249,830	17,197	12,700*
Totals..	£4,329,527	£57,418	£21,500

* Of this amount £5,000 was re-assured.

PART II.

Valuation and Bonus.

(1.) *Rate of Mortality Assumed.*

The Carlisle Table was selected as the basis upon which the Company's Premiums should be calculated, not because it was deduced from an exten-

sive collection of facts, nor because the rate of mortality furnished by it would be likely to represent that of Assured Life generally, but principally because it yielded a fair average of results, and possessed the incidental advantage of having in connection with it a large number of auxiliary tables useful for the calculation of Premiums on Special Assurances.

It will be seen from the following Table that the expectation of life, according to the Carlisle for persons under 40 years of age, has been confirmed by Mr Morgan, from the experience of the Equitable Society, and by Mr. Downes from that of the Economic; but the observations of these gentlemen and the Tables of the Combined Experience, the Amicable Society, and the Eagle Office—all Tables of authority—indicate that the Carlisle gives too favourable an estimate of the duration of life for persons above the age of 40.

TABLE.

MEAN DURATION OF LIFE.

Age.	Carlisle Table.	ASSURED LIFE.				
		Equitable (Morgan).	Economic.	Experience of 17 Offices.	Amicable, 1851.	Eagle.
20	41·46	41·67	41·40	41·49	(Not given)	37·99
25	37·86	38·12	38·57	37·98	38·86	34·98
30	34·34	34·53	34·82	34·43	34·84	31·89
40	27·61	27·40	27·20	27·28	27·11	25·54
50	21·11	20·36	19·96	20·18	19·83	19·37
60	14·34	13·91	13·83	13·77	13·32	13·52

When recommending the adoption of the Carlisle Table as the basis for the calculation of the Premiums, I was influenced also in some measure by the experience of other Offices that a large number of the Policies effected would remain on the books for a few years only, and that the mortality of the Assured would for some years after selection be considerably below the rate furnished by the Carlisle or any other Table.

But if it be assumed that the Carlisle Table represents the average rate of mortality under Policies throughout their currency, and if the actual rate immediately after the Assurances are effected is considerably below the tabular one, it follows that the mortality, after the effects of selection have disappeared, will exceed that furnished by the Carlisle, in order to make the Table true for the whole duration of the Policies. I decided, therefore, to recommend you to authorise for the purpose of the valuation the adoption of a different rate of mortality from the Carlisle, * * * a Table of Mortality which required the greatest Reserve, and guaranteed, in the highest degree, the safety and future prosperity of the office.*

* The Carlisle Table, I am aware, has been very widely adopted by Companies in their valuations ; and I observe that Mr. Thompson, the Actuary of the Standard Life Office, has recently stated that his Company had adhered to this Table for 30 years, and that the leading Offices had, one by one, been led to adopt it. I venture, however, to predict that, one by one, those Offices which have accepted it as the basis of their calculations will reject it (as an *unsafe* Table) when they find the new business in smaller proportion to the existing business at future valuations.

(2.) *Rate of Interest Assumed.*

The rate of Interest assumed in the calculation of the Premiums was 3 per cent. only, and as the same assumption has been made in the valuation, new Assurances will be placed under no disadvantage compared with existing Policies, as the liability under the latter will have been amply provided for. I have already stated that the average rate of Interest obtained by the Life Branch upon the monies invested is £4 16s. 6d. per cent. per annum.

It is not uncommon for Offices, in their valuations, to assume Interest at the rate of 3½ per cent., and, in some cases, even 4 per cent. has been assumed, though the calculation of the Premiums was based upon 3 per cent.; but it is obvious that the adoption of a higher rate in the valuation indicates a diminution in a Company's resources for meeting outstanding obligations, as, of course, a much smaller Reserve is thus provided than would have been provided had a lower rate of Interest been assumed.

The following example may be adduced in illustration:—

Several Offices, by adopting a 3 per cent. Valuation Table, reserve for their liabilities from £2,000,000 to £5,000,000. Now, if the rate were assumed at 4 per cent., the Bonus immediately following the valuation would certainly be augmented by 10 per cent. of this Reserve, or from £200,000 to £500,000; but, as a consequence, the Bonuses at subsequent divisions would be diminished, if they did not entirely disappear.

(3.) *Margin for Expenses of Management, and for forming a Bonus Fund.*

In the present valuation the whole of the "loading" has been reserved, and only that part of the Premium which provides for the risk has been valued as an Asset.

(4.) *The Surplus ascertained, and Profits to be declared.*

The Tables used by Actuaries for the valuation of the Capital sums assured, assume that they will be paid at the end of the year in which the Lives fail, or, on an average, six months after death. As, however, most Offices agree to pay the amount assured three months after proof of death, the Tabular calculations yield a lower value than the true one, and, consequently, require adjustment. In this Office a greater adjustment than ordinary is necessary, on account of the Company's Claims being payable one month after death has been proved. The amount of this adjustment, or sum to be added to the net liability, is £6,279; and a further adjustment is requisite to meet the cases where an excess of Premiums is paid in the second half of each year instead of the payments being equally distributed throughout the year, as the Tabular calculations assume. These two adjustments amount together to £9,756, and being added to the net liability under Policies and Annuities produce the sum of £129,694.

The Balance-Sheet shows that the Assets of the Life Branch, on the 31st December, amounted to .	£173,501	0	0
Deducting the total liability above-stated of .	129,694	0	0
There remains . . .	£43,807	0	0

as a divisible surplus, after assuming the Combined Experience Mortality with Interest at 3 per cent., and making the most ample provision for all

special contracts and Annuities.* Of this surplus I recommend that £41,000 be divided, leaving an additional small sum to be added to the large Reserve already provided.

One-fifth of this £41,000, or £8,200, belongs, according to the Deed of Settlement, to the Shareholders, and four-fifths, or £32,800, will be appropriated to the Assurances at present in force. This £32,800, if applied as a uniform Reversionary Bonus to all the Policies, will, on an average, yield a Bonus of rather more than £2 2s. per cent. per annum on the sum assured from their commencement, equivalent to something more than 65 per cent. of the Premiums paid.

The Surplus stated above leaves untouched a "loading" of the Premiums of £7,631 per annum, the present value of which is £118,936, and this may be taken as the measure of the provision made for future expenses and Bonus, exclusive, however, of the profit to be derived from a higher rate of Interest on the accumulated Premium Fund than the 3 per cent. assumed.

In the Valuation Summary, given in the Appendix, will be found full particulars of the value of each class of Policies, but the results may be conveniently summarised in the following form:—

SUMMARY VALUATION TABLE.

	No. of Policies.	Value of Sums Assured.	Value of Net Premiums Payable.	Difference, being Liability.	Value of Margin.
1. **Assurances.**		£	£	£	£
Participating Policies	1,547	548,279	478,223	70,056	118,588
Non-Participating „	368	183,353	159,047	24,306	16,890
Special „	55	..	..	36,772	..
	1,970	731,632	637,270	131,134	135,478
Deduct Re-Assurances	..	103,765	85,482	18,283	16,541
		627,867	551,788	112,851	118,937
2. **Annuities.**					
Present and Deferred	10	..	...	7,087	...
Adjustments . .	..	..	...	9,756	..
	..	627,867	551,788	129,694	118,937

(5.) *Distribution of the Surplus among the Policy-holders.*

The methods employed by Life Offices in the distribution of Profits are as remarkable for their number as for their diversity.

The only principle which appears to me equitable in the distribution of the Surplus is that which makes an allotment in proportion to the contribution of each Policy to the Profit Fund, and upon this principle the Profits ascertained by the present valuation will be apportioned to the Policy-holders.

* The Annuities, it should be observed, have been calculated by the Office Tables—that is to say, the Reserve made for those outstanding is the same as that at which the Office would be prepared to grant Annuities of the same amount at the advanced ages of the Annuitants.

(6.) *Modes in which the Assured may receive their Bonus.*

It appears desirable that, in distributing the Surplus to the Assured, their circumstances or convenience should be consulted, as much as possible, by the offer of several options of receiving their share, and it is, therefore, proposed that the Bonus shall be declared in the three following *reversionary* methods:

(*a.*) an immediate addition to the sum assured, to be payable with it at death.

(*b.*) the addition of a very large Bonus to the sum assured, to be payable with it at death, provided the Assured have then attained the average expectation of life of persons of his age at entry.

(*c.*) the application of the Bonus in making the sum assured payable like an ordinary Endowment Policy in the life-time of the Assured;

or that Policy-holders shall have the right to surrender their Reversionary Bonuses for

(*d.*) a cash payment; or

(*e.*) a permanent reduction of the Annual Premiums.

The election of one of the five preceding methods of receiving the Bonus, must be made before or on the 1st July next. It should be understood that if a Policy-holder decide to receive a cash payment or have his Premium permanently reduced, the Bonus allotted to him at the next division of profits will be declared upon the original amount only of his Policy; but that if he accept any one of the three reversionary methods he will, at the second distribution, receive a larger Bonus than if he had selected a cash payment or reduction of premium, as the Bonus will be calculated upon the original amount of the Policy and the sum by which that Policy has been augmented at the present division.

TABLE.

Showing the approximate amount of Bonus upon a Policy for £1,000 *effected in the year* 1863.

Age at Entry.	Present Age.	REVERSIONARY BONUSES. Immediate.	Deferred. Age.	Deferred. Amount.*
		£ s. d.		£ s. d.
20	25	104 0 0	61	272 0 0
30	35	104 10 0	64	259 0 0
40	45	105 0 0	68	257 10 0
50	55	110 0 0	71	253 0 0
60	65	121 10 0	74	235 0 0

* Payable with the sum assured, provided the Assured attain the average expectation of life of persons of his age at entry. By this plan a large addition to the Policy is made at each successive division of profits.

Summary and Valuation of the Policies of the Commercial Union Assurance Company, as at 31st December, 1867.

Description of Transactions.	No. of Policies.	Sums Assured.	Office Yearly Premiums	Net Yearly Premiums.	Net Liability.*
Assurances.					
1. WITH PARTICIPATION.		£	£	£	£
For the Whole Term of Life	1,469	1,088,440	34,657	27,900	65,082
Do. (Premiums by a limited number of payments)	13	12,550	963	805	2,491
Do. (Increasing Premiums)	7	5,800	137	107	141
Endowment Assurances	25	8,000	361	290	491
Endowments	1	2,000	69	58	413
Deferred	15	31,500	688	535	1,069
On Two or more Lives	17	7,200	255	194	369
Assurances with Profits	1,547	1,155,490	37,130	29,889	70,056
2. WITHOUT PARTICIPATION.					
For the Whole Term of Life	249	256,820	9,481	8,593	20,136
Do. (Premiums by a limited number of payments)	1	800	19	17	89
Do. (Increasing Premiums)	50	43,210	828	718	933
Endowment Assurances	32	25,893	2,148	1,980	2,177
Joint Lives	15	7,450	301	261	281
On the Longest of Two or more Lives	10	7,570	150	130	382
Deferred	1	1,000	9	7	26
Children's Endowments	10	1,384	45	42	282
Policies not comprised in the above Class (Including Assurances for Terms of Years, against the Contingency of Issue, Contingent Survivorships, Endowments at short dates, including one for a large amount due in 1868, &c. &c.)	55	143,291	3,950	3,555	35,458
Extra Premiums payable	..	..	951	.	1,315
Assurances without Profits	423	487,418	17,882	15,303	61,079
Total Assurances	1,970	1,642,908	55,012	45,192	131,135
Deduct Re-Assurances	117	222,874	8,302	7,064	18,283
Net Amount of Assurances	..	1,420,034	..	..	112,852
Adjustments, to provide for the payment of Policies one month after proof of death	..	..	..	..	6,279
Do. to equalise the excess of Premiums payable in the second half of any year	..	..	..	..	3,477
Annuities.		per annum			
Present and Deferred	10	906	..	..	7,087
Total of the Results			**46,710**	**38,128**	**129,695**

* Value by the Combined Experience Tables, Interest 3 per Cent.

Consolidated Account of the Receipts and Expenditure of the Life Branch, from 1st May, 1862, to 31st December, 1867.

1867.		£ s. d.	£ s. d.
Dec. 31.	To New Premiums, including Single Premiums, and Premiums by a limited number of Payments	74,279 16 8	
	„ Renewal Premiums	148,352 0 9	
			222,631 17 5
	„ Consideration for Annuities		7,548 18 10
	„ Bonus on Re-Assurances		42 1 0
	„ Fines for Extension of Time		165 4 4
	„ Interest		15,748 9 1
			£246,136 10 8

1867.		£ s. d.
Dec. 31.	By Claims	11,300 0 0
	„ Surrenders	2,103 10 0
	„ Re-Assurances	36,714 16 9
	„ Annuities	3,575 8 9
	„ Commission, *less* that received on Re-Assurances	8,243 1 7
	„ Expenses of Management	14,576 5 0
	Balance	169,623 8 7
		£246,136 10 8

Balance Sheet of the Life Branch of the Commercial Union Assurance Company on 31st December, 1867.

LIABILITIES.

I. Amounts due by the Life Branch—	
1. Claims by Death, not yet payable	£5,100 0 0
2. Commission due to Agents	1,227 13 2
3. Re-Assurance Premiums	1,102 18 4
II. Value of Liabilities under Assurance and Annuity Policies, per Summary Valuation Statement	129,694 0 0
Balance	43,807 15 0
	£180,932 6 6

ASSETS.

Loans on Mortgages	£116,311 0 4
„ „ Policies	5,475 3 1
Great Southern of India Railway Guaranteed Stock	6,026 5 7
£4,600 Canada 5 per Cent. Bonds	4,134 5 0
Miscellaneous Sums due to the Life Branch	48,985 12 6
	£180,932 6 6

GRESHAM LIFE ASSURANCE SOCIETY.

Established 1848.

General Report of the Directors.

The Extraordinary General Meeting of the Gresham Life Assurance Society held this day, (14th Nov., 1867,) having empowered the Directors to divide profits at shorter intervals than every five years, the Board of Directors have determined that a division of profits shall be made amongst the Members in respect of surplus funds existing on the 31st July, 1867, and it has been resolved that the future divisions shall take place at intervals of three years, until otherwise determined.

The Directors having instructed their Actuary to make a very careful investigation of the assets and liabilities of the Society as existing on the 31st July, 1867, have now to report thereon.

As regards the Society's assurance and annuity contracts, the groundwork for the valuation of them was the table of mortality known as the Experience of the seventeen offices, and the rate of interest assumed in the valuation 4 per cent. The former being that which, more nearly than any other table, represents the rate of mortality amongst the assured lives in the "Gresham," and the latter being below the average rate of interest actually obtained on the invested funds of the Company, which, being very nearly four-and-a-half per cent. is about one quarter per cent. more than the average rate obtained at the period of the valuation in 1865.

The investigation made by the actuary of the assurance and annuity contracts shows that there were in force on the 31st July, 1867, 19,634 policies for assurance under various contingencies of life. The present value of the sums assured was £3,590,461 19*s*., and the present value of the prospective income from premiums was £3,395,658 2*s*.

Also that there were in force at that date 365 policies for annuities, either immediate, contingent, or deferred, the present value of which was £100,723 14*s*.

From the general Balance-sheet of assets and liabilities appended to this report, it will be seen that the balance in favour of the Company is £665,591 12*s*. 8*d*. This amount constitutes the fund, out of which the present bonus and future profits are to be appropriated, and the future expenses are to be paid.

In 1860 and 1865, the policies then existing had been in force 3½ years on the average. The average duration of the policies in force on the 31st July last was rather over 4 years.

The attention of the members was drawn in 1864 to the proportion existing between the realised assets of the Company applicable to the contracts of assurance, and the amount of premiums received on the policies then in force. At that time the proportion was 53 per cent.: it is now 70 per cent.

After a careful consideration of the whole facts, and actuated by prudence and caution on the one hand, and by regard for the interests of the policy-holders on the other, the Directors have determined to apportion amongst the members, by way of bonus, the sum of £60,000.

The Company's income, prospective balance, and realised assets at the periods of the several valuations made of the Company's affairs, are shown in the following tabular statement:—

Valuation. Years.	Income.	Prospective Balances.	Realised Assets.
1852	£23,141	£74,478	£49,662
1855	43,248	105,273	119,377
1860	108,226	206,122	230,166
1865	223,423	312,933	760,796
1867	297,699	665,591	1,025,482

Eighty per cent. of the bonus declared belongs to the participating policy-holders, and will be applicable to all the participating policies in

force on the 31st July last. The bonus applicable to each particular policy will be calculated forthwith so that it may be appropriated on the 30th June next.

In the case of claims under participating policies by death or endowment happening prior to the 30th June, 1868, the cash value of the bonus to be allotted to such policies will be allowed.

In other cases, the usual options will, on the 30th June next, be afforded to the policy-holders in selecting the modes in which the bonus should be applied.

GENERAL BALANCE-SHEET for 31st July, 1867.

Assets.	£	s.	d.
Present Value of Policy Premiums, and Reassurances	3,401,463	5	0
Investments in Funded, Freehold and Leasehold Property . .	456,525	12	9
Mortgages on Real and Personal Estate	340,594	14	0
Loans on Policies on Credit Premiums, etc.	70,527	16	10
Cash, Bills, Bankers' and Agents' Balances, and Current Premiums.	157,833	19	5
Current Interest on Investments	8,318	18	4
	£4,435,264	6	4
Liabilities.			
Value of Sums Assured and Bonus	3,590,461	19	0
Value of Annuities	100,723	14	0
Sundry Claims and Charges waiting Settlement	56,775	0	8
Proprietors' Fund	21,712	0	0
	£3,769,672	13	8
Balance available for present and future Profits and future Expenses	£665,591	12	8
	£4,435,264	6	4

NOTICES OF NEW WORKS.

The Insurance Guide and Hand Book: being a Guide to the Principles and Practice of Life Assurance; and a Handbook of the best Authorities on the Science, &c. By Cornelius Walford, Barrister-at-Law, Fellow of the Statistical Society of Great Britian, &c. &c. Second Edition. C. & E. Layton.

This book has established its *raison d'être* in the way of all others most pleasing to an Author. A second Edition is at all times something more than a compliment; in a case like this, it is an evidence of marked, and, we would add, deserved success. A bulky volume of over 400 pages and of proportionate cost, bearing the somewhat forbidding title of "The Insurance Guide and Handbook," and, though dedicated especially to Insurance Agents, having in reality no natural audience, would at first sight seem little likely so completely to exhaust "a first Edition of some Thousands" as to have been "long out of print" ten years after its publication. Yet such is the fact here, and hence this new issue.

On reading the book, it is not difficult to divine the sources of its popularity. The Author himself says, at page 65, "Writing for a popular purpose, and on a subject which certainly *ought* to be popular, we freely

admit that we have not limited ourselves to such points as Insurance Agents *only* should understand. Acting upon the belief that all that relates to humanity relates to every man, we have assumed a broader basis, and endeavoured to bring together, in these pages, a store of facts and considerations which every man may be the wiser and better for becoming acquainted with." We are accordingly prepared to find much that bears but remotely on the main object of the work, which was to supply "a Guide to the principles of Assurance" and "a Handbook on essential points of practice." The second division, of 85 pages, devoted to "popular notes on the laws of mortality, as bearing upon Life Assurance; also on population and longevity," is of a character especially gossiping. In it are mixed up with facts of undoubted value, such merely curious or amusing matters as causes of death in Royal Circles, length of reign of European monarchs, the voyage of life, marriage and its influences, the hours most fatal to life, suicide, the art of prolonging life, the three celestial gifts—in this case, light, heat and air, longevity of the patriarchs, and the art of enjoying health. There is, however, a more solid reason for the success this book has met with—a success beyond that at which it may be said to have aimed. Many persons who have little to learn from its pages are yet glad to avail themselves of it, as being, in an honourable sense, a kind of *omnium gatherum* of facts and opinions, conveniently grouped, drawn chiefly from writers of authority, to whose works reference is always made with scrupulous fidelity. Encyclopœdias and vade-mecums will always be dear to that numerous class who dislike the labour of original research.

The book comprises six divisions, including the one just spoken of; and although it is beyond our scope to enter into any elaborate analysis of their contents, it may be useful to give their titles and to describe briefly their subjects, the more so that the book, although containing a full index, has no Table of Contents.

Division I. contains an account of the early History and progress of Insurance—Marine, Casualty, Fire, and Life—and may be shortly described as an attempt to trace the progressive development of the Insurance principle from its germination in the very dawn of our modern civilization, through its various inchoate and speculative stages, to its luxuriant growth in that era of spasmodic energy, the reaction from which may be seen in the melancholy list of abortions and decays of nature, or, in plain words, of unsuccessful Companies, with which the chapter ends.

Division II., besides those merely amusing matters already noticed, deals with the really important subjects of mortality, in reference both to age, condition, and locality; of deaths, both as to their ordinary causes and their fluctuations with the political condition of the country—with its years of peace and plenty, of war and scarcity.

Division III., which treats of the theory and practice of Life Assurance and commences at page 149, is the real beginning of the serious purpose of the book. Having traced with much elaboration, interest and success, the history and characteristics of the various tables of mortality, which, as the author says truly, but with some confusion of metaphor, are "the key-stone or pivot upon which the whole science of Life Assurance hinges," and having traced also the history, if the expression may be allowed, of the rate of interest, and its bearing on the financial operations of Assurance Companies,—that is, having treated of "the two fundamental elements of Lives and money, upon which," the author again says truly, but this time

with some confusion of grammar, "the structure of Life Assurance has, and must, continue to be raised,"—he discusses, with both fairness and clearness, Life Assurance in its modern practice and in regard to the improvements and expansions of which he thinks it capable; and then ends, somewhat oddly, with a chapter on "rates of premiums; method of determining them; and tables of rates of all the existing Offices."

Division IV. relates to "Bonuses: how derived, and the proportions and manner of declaring them"; and we are glad to bear testimony to the care with which the Author states and enforces the sound and judicious views of various writers on the vital subjects of valuation and division of apparent surplus.

This division concludes with a "Bonus Table," the object of which is very laudable; but which, by attempting too much, has become positively misleading. The Table includes columns giving the date of the first division of profits, the proportion given to the assured, and how often the profits are divided; and these facts, although subject to constant changes, have a degree of permanent value, as showing the practice of the Offices at a given time. But the facts given are not accurate, even as relates to so simple a matter as the proportion of the profits returned to the assured. For example, when it is stated in the case of one Office that two-thirds only of the profits are allotted to the assured, nothing is added to remind the reader that the profits here meant are the *gross profits*, before any deduction is made for the expenses of management; and that the two-thirds of these gross profits will be a much larger proportion of the net profits. Nor is there any intimation given that in many instances the assured receive the stated proportion of the profits *from every source*, while in others they only receive a proportion of the profits arising from the participating policies—the difference often being very material. There are many obvious errors and inconsistencies in the column "amongst whom the profits are divided"; but the next one, "Principle of Division," is so very inaccurate, that we are forced to the conclusion that the statements it contains must have been arrived at by pure conjecture. To give this information with the slightest approach to accuracy would require a long chapter to be devoted to the subject; and even if this were done, the information would be of very doubtful value, on account of the constant changes which are always taking place in the methods of division of profits. Notwithstanding all that has been written on the subject, we are of opinion that no complete and impartial account has yet appeared of the various methods of division of surplus actually in use; and such an account is much to be desired. Lastly, in the column "How the profits are applied," there is no hint given that several Offices, which are returned as allowing the option of addition to the sum assured or reduction of premium, require the option to be exercised on one occasion only—at the date either of effecting the policy, or of the first division of profits—while others allow the option to be exercised at every division of profits; and some again allow the reversionary bonuses to be surrendered *at any time* for the equivalent value in cash or reduction of premium. On the whole, it would be perhaps better that this table should be entirely omitted. If retained in a future edition, it should certainly be re-cast.

Division V. treats of "Life Assurance as an investment, including the purchase of Life Policies, and speculations in Insurance Shares," and embodies various annuity and other tables, besides one which is open to

even greater objection from the evanescent character of its facts than the table concerning Bonuses just noticed—viz., a table of the Capital of Insurance Companies and of the "present price" of their Shares! Life Assurance as an Investment is a subject which has never been treated in the serious and exhaustive manner its importance deserves. The Author in his Chapter on this question gives a table comparing the position of two men of the age of 30,—one of whom accumulates £5 per annum at compound interest, whilst the other invests the like Annual Sum in insuring his life for £200, with profits—during each of the 34 years which measure their "Expectations." To the one he allows 4 per Cent compound interest, and to the other a fixed and not immoderate rate of bonus arising quinquennially; and the comparison so brought out shows a balance, large at the beginning and, though of course gradually diminishing, continuing in favour of the Policyholder during the whole of the period, excepting the last 4 years. Now, regarding this statement as that of an account closed by death, which is of course the assumption, it is conclusive enough so far as it goes. But it unfortunately stops at the very point where the difficulty begins, *i.e.* when the accumulated premiums begin to exceed the amount payable under the Policy. The merits of Assurance are clear enough to those who receive the proceeds of early claims; the long livers alone can be the grumblers, and it is with them that the controversy has to be carried on. By stopping where it does, this table gives a colour to the notion which these commonly hold, that when more has been paid than can be received, the Insured has made a bad investment. It is unnecessary to say here how wide this is of the truth; the only question is as to how it is to be shown that it is not true. The difficulty is to make men see, or rather perhaps acknowledge, after the event, that they are properly made to pay for the risk of a contingency which has not happened. With a yearly fire risk, or accidental risk, no such difficulty is experienced; nor would any probably be felt with regard to life assurance, if what may be termed the *natural system of assurance* were the custom, that is, if the current risk only of each year were paid for by the variable premium necessary to cover it, in which case, we may observe in passing, life insurance would cease to be an "investment." It would then be seen by the most obtuse person that at the end of the year he had had the full equivalent for his money, even though the event for which it was paid had not happened. Hence we should be glad to see this Table differently constructed. The most intelligible plan would perhaps be that which should show year by year the accumulation of the natural year-to-year premium for the difference between the amount of the assurance and the amount of the accumulated investment. When, for example, these balance each other, as in the table they do practically at the end of 31 years, it is not accurate to say that then the two plans have been equally advantageous, and still less, as the Author does, that from that point the advantage is in favour of the ordinary investor. Surely something is due for the guarantee during all these years for the payment in case of death of sums ranging from £195 in the first year to £12 nearly in the 30th! If the value of such guarantee be either added to the amount of the assurance, or, still better, deducted from the amount of the accumulated investment, the balance would be at once destroyed. If the same plan were followed in subsequent years, the fact would be apparent, which nobody who understands the subject will doubt, that except in the most extreme cases Life Assurance can never be a bad investment.

Division VI. is a miscellaneous one, having chapters on the selection of an Office by an Agent or by an intending insurant, on the working of Assurance Agencies, and on the publication of accounts and the legislation needed for Assurance Offices, about which we would make but this remark, that never more than now has interference on the part of the state been needed, in order to enforce such a measure of publicity as would enable persons of ordinary intelligence to judge of the solvency of the Companies to which they have committed their interests. The state of things existing with regard to Life Assurance Companies has no parallel. The public have by the nature of things to repose in them great and, in theory at least, perpetual confidence; and they, in turn, too often treat the public with the most studied and persistent secresy. None but Companies that fear the light would do other than welcome an impartial parliamentary action which, whilst leaving them unfettered in the conduct of their business, would compel them to state its results with such clearness and precision as would at least rob dishonesty of its congenial darkness.

On all the subjects embraced in these divisions, little, whether of theory or of practice, is omitted from consideration; and the Author rarely departs from his custom, indicative at once of modesty and good sense, of supplementing his own remarks with copious quotations from the writings of others. It is however one of the misfortunes incident to a second Edition of a work like this, that many of the authors quoted as authorities have by lapse of time ceased to be such, even if the title fairly belonged to them in the first instance, as it is that no reference is made to the writings of the many able men who have in the interval between the two Editions brought themselves to the front.

Indeed, we have throughout our perusal of the book been constantly reminded of the little that has been done to bring its facts and opinions down to the present time. We are told, for example, at page 199, that "The Amicable still remains secure in its antiquity," and, at page 309, that "the Family Endowment and the Britannia apply their Bonuses to reduction of premiums only," though elsewhere these Offices are all properly treated as non-existent. Again, at page 172 we are told, on the authority of Mr. Sang, but in spite of the researches of Messrs. Bailey and Day, that a greater share of health and longevity "is enjoyed by the working population (the middle classes generally) than by those who have been more favoured with the smiles of fortune." We still read in numerous places of *Dr.* Milne and *Dr.* Griffith Davies, misnomers, speaking of unacquaintance, that ought not to have escaped any kind of revision, whilst there remain all the old printer's mistakes, of which the whole book is more than enough full. In a second Edition, one would scarcely expect to see it gravely stated, as it is on page 212, that the Globe Office "invested its Capital in the funds at a period when they were low, and thereby obtained very nearly *two millions* for *one*," or that in Banks, unlike other Joint Stock Companies, "the liability of the Shareholders is unlimited." The Chapters on population, &c., in Division II. have confessedly not been revised on the last Census returns, the results of the former Census being in the opinion of the Author sufficient for his purpose. If, however, it be his purpose to leave on the mind intelligent ideas of facts as they now are, we take leave to differ from him. It is manifest that his figures relating to the population have lost all their original value, except as mere historical numbers. They no longer refer to an existing or even a proximate state of things. In a modified

sense, the same may be said of those relating to mortality. It is no longer accurate, for example, to say that the mortality of London is at the rate of 25 per 1000, nor that that of England and Wales is at the rate of 23 per 1000, both rates having undergone appreciable diminution in recent years.

We are reminded by what we have now said about the vanished trustworthiness of these figures, that even on their own basis they are not always to be depended on. To take two or three examples. At page 73, the mortality of London is given as 1 in 41; at page 75, as 1 in $20\frac{3}{4}$; and at page 117, of 1 in 39. Again, at page 82 the deaths in childbed are stated as 1 in 171 births registered, whilst at page 139 they are given as 1 in 200 children born alive. And again, at page 130, "physicians who practise" are said to be "shorter lived than almost any other of the professional classes," although on the very next page there is quoted Dr. Guy's dictum that medical men occupy the first position, in respect of longevity, of the three learned professions.

There are other blemishes of a verbal and numerical kind. At page 56, in a table of the number of Companies formed under the Act of 1844, the additions of two columns of figures that really add up to 49 and 243 respectively are printed as 27 and 258; at page 166, in the comparative table of "Expectations," the expectation at age 65 by the Equitable Experience is given as 19·4 instead of 11·4; and at page 176, in a table of decrements according to the English life tables No. 1, the number of males who die in the 6th year of life out of 512 born is given as 56 instead of 5. Again, at page 358, £397 is stated to be the value of a reversion to £1000 on the death of a life aged 53, the 53 being a mistake for 50; and at page 334 there is an error of like kind, the value of a reversion to £1,000,000 at age 30 being stated as £446,750, whereas, by the table of mortality and at the rate of interest assumed, it is the value of such a reversion at age 35. These are examples which have mostly caught the eye and are by no means the fruit of diligent search.

The Author does not write as an actuary, and is, therefore, not to be judged by the severe rule which would rightly be applied to one having more pretensions. But we notice some expressions which are something more than mere verbal errors, and which should have been corrected in a second edition. Thus, at page 153, the term "mean duration of life" is defined as if it were applicable to the period of birth only, the Author seeming to be unaware that it is another expression for the "expectation of life." Again, the explanation, at pages 151–2, of the graduation of mortality tables is very imperfect, and the illustration from the Northampton Table unfortunate, the error in the construction of that Table being one of principle, which cannot be remedied by graduation. At page 251 the supposed use of the term of the expectation of life in the construction of premiums is altogether wrong; at pages 303–4, in describing the mode of distributing the Bonus as a percentage on premiums paid, the cash allotment is treated as the Reversionary addition that would be made to the Policies; and lastly, at page 355, in dealing with modes of valuing a Policy for purchase as an investment, the Author gives as a correct method one by which the difference between the *gross* premiums at the time of purchase and at entry is multiplied into the annuity, which, it is obvious, is correct only when the purchaser intends to effect an insurance on the life in question, but has the opportunity of purchasing an existing policy.

We should be sorry to end here. The defects we have pointed out are after all but specks in the sun. Though they sometimes dim, they never

obscure, the substantial merits of the book—its comprehensiveness, completeness, fairness and soundness. The Author shows himself ever jealous for the honour of Assurance Institutions, and always, in intention, loyal to their best interests. He is earnest alike in his denunciation of irregularities within and in his exposure of delusions without, as well as honest in his eager preference for that sober moderation which in all things puts aside a present, if a doubtful, good, for a permanent, though deferred, success. The cause of Assurance cannot be the worse, it ought to be greatly the better for this work. How much it has been bettered in fact, it would be exceedingly difficult to say. After all the labour of so many able and honest men, and all the accumulated experience of both the true and the false in the practice of Assurance, it is at once a matter for surprise and mortification that there should have been brought about so small an abatement of public credulity; but, though matter for mortification, it is no matter for surprise that, the credulity remaining, official malpractice, designed to impose on it, should remain also.

CORRESPONDENCE.

ON THE VALUE OF A POLICY ON THE LONGEST OF TWO LIVES.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—It has been demonstrated in the *Journal*, vol. xi., page 104, that the Value of the Policies existing in an Office is equal to the accumulated net Premiums less the claims. The application of this result to an ordinary Policy on a single life is shown in the formula $\frac{N_{x-1}-N_{x+n-1}}{D_{x+n}}\cdot\frac{M_x}{N_{x-1}}-\frac{M_x-M_{x+n}}{D_{x+n}}$, vol. xi., page 107, where $\frac{M_x}{N_{x-1}}=\varpi_x$. This formula will also apply, by simply changing the values of M, N, D, and ϖ, to Policies of some other descriptions, viz., those where no alteration takes place in their status till their expiry or till claims arise on them, as for instance, Policies on Joint Lives, Endowment Assurances and Term Policies. But the formula will not be applicable, at least by a like simple modification, to the case of Policies, the status of which may be changed during their continuance, as that of Policies on the Longest of two lives, which may be changed by the dropping of one life.

I have thought a similar demonstration in the case of a Policy on the Longest of two lives to the one above referred to might not be without interest.

Let there be l_x lives of the age x and l_y of the age y, then the number of pairs which can be formed out of them, each containing a life aged x and a life aged y, is $l_x \times l_y$. Let this be expressed by $l_{x.y}$, so that in the following demonstration $l_{x.y}$ signifies $l_x \times l_y$. Let each of these $l_{x.y}$ pairs be assured for £1 payable on the last death, at an annual net Premium ϖ. Then the number of claims (that is, of pairs in which both lives become extinct) in the first year, will be $l_{x.y}-l_{x.y+1}-l_{x+1.y}+l_{x+1.y+1}$; and the sum reserved at the end of the first year for existing Policies will be

$$l_{x.y}\varpi(1+i)-(l_{x.y}-l_{x.y+1}-l_{x+1.y}+l_{x+1.y+1}) \qquad (1)$$

But of the latter Policies there will be $l_{x+1.y}-l_{x+1.y+1}$, by which only sur-

viving single lives aged $x+1$ are assured, and $l_{x.y+1}-l_{x+1.y+1}$, by which only single lives aged $y+1$ are assured. The value of these Policies will be

$$(l_{x+1.y}-l_{x+1.y+1})\{1-(d+\varpi)(1+a_{x+1})\}$$
$$+(l_{x.y+1}-l_{x+1.y+1})\{1-(d+\varpi)(1+a_{y+1})\} \qquad (2)$$

which deducted from (1) will give the value of the Policies on the $l_{x+1.y+1}$ surviving pairs of lives.

The investigation of this for the first year is as follows:

In $(l_{x+1.y}-l_{x+1.y+1})\{1-(d+\varpi)(1+a_{x+1})\}$

$$l_{x+1.y}(1+a_{x+1})=l_{x.y}\cdot\frac{l_{x+1}}{l_x}(1+a_{x+1})=(1+i)l_{x.y}a_x$$

$$\therefore \quad -l_{x+1.y}\{1-(d+\varpi)(1+a_{x+1})\}=-l_{x+1.y}+(1+i)l_{x.y}(d+\varpi)a_x$$

and $\quad -l_{x.y+1}\{1-(d+\varpi)(1+a_{y+1})\}=-l_{x.y+1}+(1+i)l_{x.y}(d+\varpi)a_y$

The sum of these is

$$-l_{x+1.y}-l_{x.y+1}+(1+i)l_{x.y}(d+\varpi)(a_x+a_y)$$

and the other terms of (2), with their signs changed, are equal to

$$l_{x+1.y+1}(A_{x+1}+A_{y+1})-l_{x+1.y+1}(2+a_{x+1}+a_{y+1})\varpi.$$

Again, $\quad A_{x+1.y+1}l_{x+1.y+1}=A_{x.y}l_{x.y}(1+i)-(l_{x.y}-l_{x+1.y+1}) \qquad (a)$

and $\quad (a_{x+1.y+1}+1)l_{x+1.y+1}=a_{x.y}l_{x.y}(1+i) \qquad (b)$

If all the terms of (1), and (2) as above modified, are collected and $A_{x+1.y+1}l_{x+1.y+1}$ and $(a_{x+1.y+1}+1)l_{x+1.y+1}$ are added and subtracted according to equations (a) and (b), the result is

$$-l_{x.y}+l_{x+1.y}+l_{x.y+1}-l_{x+1.y+1}-l_{x+1.y}-l_{x.y+1}$$
$$+(1+i)l_{x.y}d(a_x+a_y)+(1+i)l_{x.y}A_{x.y}-l_{x.y}+l_{x+1.y+1}$$
$$+(1+i)l_{x.y}\varpi+(1+i)l_{x.y}(a_x+a_y)\varpi-(1+i)l_{x.y}a_{x.y}.\varpi^*$$
$$+l_{x+1.y+1}(A_{x+1}+A_{y+1}-A_{x+1.y+1})-l_{x+1.y+1}(1+a_{x+1}+a_{y+1}-a_{x+1.y+1})\varpi.$$

Now $l_{x.y}=(1+i)v\,l_{x.y}$ and $A_x=v-(1-v)a_x=v-da_x$

$$\therefore -2l_{x.y}+(1+i)l_{x.y}d(a_x+a_y)+(1+i)l_{x.y}A_{x.y}$$
$$=-(1+i)l_{x.y}\{2v-d(a_x+a_y)-A_{x.y}\}$$
$$=-(1+i)l_{x.y}(A_x+A_y-A_{x.y})$$

Also $\quad \varpi=\dfrac{A_x+A_y-A_{x.y}}{1+a_x+a_y-a_{xy}}$

$$\therefore \quad (1+i)l_{x.y}(1+a_x+a_y-a_{x.y})\varpi=(1+i)l_{x.y}(A_x+A_y-A_{x.y})$$

Therefore the sum of the terms in the above result before the asterisk (*) vanishes, and the remaining terms divided by $l_{x+1.y+1}$ are equal to

$$A_{x+1}+A_{y+1}-A_{x+1.y+1}-(1+a_{x+1}+a_{y+1}-a_{x+1.y+1})\varpi,$$

the formula for the value of a Policy on the longest of two lives.

I am, Sir,
Your obedient servant,
THOS. CARR.

7, *Royal Exchange,*
June, 1868.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

On the Valuation of Reversionary Life Interests. By THOMAS BOND SPRAGUE, M.A., *Vice-President of the Institute of Actuaries.*

[Read before the Institute, 30th November, 1868.]

THE valuation of reversions, absolute and contingent, is a matter of great and growing importance. The greater part of the landed property in the United Kingdom being settled in strict entail, proposals are being constantly made by the expectant heirs for loans upon their reversionary estates; and so long as the present law and practice of settlement prevail, so long will the making advances on the security of reversionary interests in landed property continue to be an important part of the business of such Life Insurance Companies as are wise enough to cultivate it. Although the amount of the landed property in the country remains unaltered, yet the rentals are steadily increasing, and consequently the amount of the money which can be employed in the way above described may also be expected to increase. Again, the vast and rapid increase of wealth among the middle classes of the country has led to a great increase in the number of life interests and reversions under marriage settlements and wills; and in consequence of the many vicissitudes in trade, such reversionary interests in personal property are constantly offered for sale in increasing numbers. There is also, perhaps, a growing disposition,

when the owners of reversionary interests in either real or personal estate become involved, to press their interests for public sale, instead of arranging privately a sale to some member of the family. On the other hand, as it is becoming better known how ready Offices of good standing are to make advances on contingent reversionary interests, applications are now made to them, which would formerly have been made to private money lenders, whose charges were, to say the least, excessive. From these combined causes, the number of reversions in the market is increasing, and seems likely to increase.

There are in existence several Reversionary Interest Societies founded for the purpose of employing the capital of the shareholders in the purchase of reversions; but these investments appear to be even better suited for Life Insurance Companies. The latter Companies, when once well established, have for a long series of years rapidly increasing assets, for which investments have continuously to be found. They can estimate within narrow limits the probable amount of the claims upon them; and they require to have only a small portion of their assets readily convertible. They can therefore afford to invest a large portion of their assets upon loans for long terms, such as mortgages of parochial rates and drainage rent-charges; and they thus obtain the advantage of the higher rate of interest which such securities produce, as compared with ordinary mortgages and the other securities for which there is a greater demand by trustees and private individuals. For the same reasons, they can afford to lock up considerable sums in the purchase of reversions, which on the average produce a still higher rate of interest than any of the securities above referred to. In the case of the purchase of an absolute reversion, it is a serious objection to an individual purchaser, that he receives no income from his investment; and in the case of a contingent reversion, there is the further objection that he has to incur a fresh outlay every year for payment of premiums; but so far from this being an objection when an Insurance Company is concerned, it is matter of congratulation when a contingent reversion, or reversionary life interest is purchased, that a remunerative investment has been made for part of the surplus assets of the future as well as of the present.

Before proceeding to the more particular object of the present paper, which is the explanation of some new formulæ for calculating the value of reversionary life interests, it will be convenient to consider the existing formulæ for the purchase of absolute and

contingent reversions. It was formerly, I believe, the universal practice to value absolute reversions offered for sale by the ordinary tables of single premiums, using such a rate of interest,—5, 6, or 7 per cent—as circumstances seemed to call for. But in the second volume of this *Magazine*, page 162, Mr. Jellicoe, in the course of a remarkably able and comprehensive paper "On the contrivances required to render Contingent Reversionary Interests Marketable Securities," proposed a new method. His formula for the value of the reversion to £1 on the death of a person, aged x years, is $1-d(1+a_x)$, where d is the discount on £1 payable at the end of a year, and is to be taken at 5 per cent interest; and $1+a_x$ is the value of the annuity-due on the life, and is to be taken at $3\frac{1}{2}$ per cent. We may thus write the formula,

$$1-d_5(1+a_x)_{3\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

How far this method has been adopted in practice by Actuaries, I am unable to say. It has been adopted by Mr. Tucker, among others, who speaks in praise of it in vol. v., pp. 166 and 242. (See also Mr. Day's paper on the Purchase of Life Assurance Policies as an Investment, vol. viii., page 326.) The only other discussion of the formula I can find is contained in Mr. Scratchley's Work on "Associations for Provident Investment." Mr. Jellicoe and Mr. Tucker urge the adoption of the formula on the ground that the value of a reversion should be such as will enable the purchaser to protect himself from all contingencies. It should be such as to allow of his purchasing an annuity that will furnish interest at five per cent on his total outlay until the reversion falls in. For it is not reasonable (Mr. Jellicoe in substance says) that the purchaser of an "isolated reversion" should be exposed to the risk of heavy loss in consequence of the tenant for life attaining extreme old age; and if he is secured against such loss by the purchase of the annuity above supposed, he can only fairly require such a rate of interest as is yielded by good mortgages. Mr. Scratchley, on the contrary, maintains that "the formula cannot be supported by any satisfactory reasoning, and is objectionable from its giving *negative* results for ages under a certain point." (See however the remarks in vol. vii., page 54.) Mr. Scratchley proceeds to recommend a new formula for the valuation of absolute reversions,

$$\frac{(\varpi_x)_5}{d_5+\varpi'_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(where ϖ_x is the net premium calculated at 5 per cent, and ϖ'_x the

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office premium, for an insurance of £1 on the life of x); concerning which it may suffice to say that it does not seem to have been noticed, much less adopted, by any other actuary; and that the examples he gives of the values resulting from its use prove it to be altogether unsuitable for practical transactions.

Anything that is recommended by the high authority of Mr. Jellicoe and Mr. Tucker calls for the most careful consideration; and any actuary who dissents from their views should be prepared to state fully and clearly his reasons for such dissent. I will therefore do my best to explain the reasons which have induced me for a long time past to abandon altogether the use of Mr. Jellicoe's formula.

First, then, when the question arises,—*What is the market value of a certain reversion?*—I object to introducing the idea of the purchase of *isolated* reversions. There is in practice a very ready market for reversions; and indeed, whenever an absolute reversion free from all objection is offered for sale by public auction, there is a keen competition among the Reversionary Companies and other purchasers to obtain it. The market price then must certainly be that price which the principal purchasers,—the Reversionary Companies—find it worth their while to give; and the question resolves itself into determining upon what principle these Companies should regulate their purchases.

This brings me to my second objection to the new formula; viz., that it supposes an annuity to be purchased to provide interest on the outlay until the reversion falls in, whereas such an annuity is never actually purchased in practice. This being the case, if the value of a reversion is estimated by Mr. Jellicoe's formula, allowing 5 per cent interest to the purchaser, while the annuity is calculated at 3½ per cent only, the purchaser of a large number of reversions, such as a Reversionary Society, would make a profit on the assumed grant of these annuities, in addition to the 5 per cent. It therefore appears to me much more satisfactory to include the whole profit of the transaction under one aspect, and to assume that the reversions are on the average bought on such terms as to yield a higher rate of interest. Assume, then, that a Reversionary Company buys its reversions at prices found from the six per cent single premiums, then if the number is sufficiently large, they will on the average fall in at such times as will return to the Society the cost together with six per cent compound interest. If the expenses of management are one per cent per annum on the capital, then the shareholders will receive five per

cent on their capital. It cannot of course be supposed, that the shareholders will be satisfied with a less return than this; and it would probably be too sanguine to assume that the expenses of these Companies can ever be permanently reduced much below one per cent per annum; and we may therefore fairly take six per cent as the proper rate upon which the market value of reversions may be estimated. There is however, I think, very little doubt that reversions dependent upon young lives may be bought to pay a higher rate of interest than those dependent upon old lives—say over 60; for this reason, that private purchasers who will often bid for the latter, avoid the former, on account of the smaller probability of their living to see the reversion fall in.

Such, then, being the terms on which the principal purchasers deal, the prices will in general be the same for other purchasers. I have said that these investments are even better suited for Insurance Companies, than for the Reversionary Companies. That this is so will appear when it is considered that an Insurance Company possessed of large funds may transact the whole business of a Reversionary Company without any increase of expense, beyond, perhaps, the employment of an additional clerk. It will not perhaps be too much to prophesy that as the Insurance Companies become more alive to the advantages of this class of business, they will drive the Reversionary Companies out of the field. In this connection it may be well to advert to an objection often raised against the purchase of reversions, viz., that it encourages young men to anticipate their future expectations. That this objection has no real weight will appear from the undoubted fact that reversions are almost always sold to pay debts already incurred; that the spending,—the real anticipation of the future,—has already taken place before the reversion is offered for sale.

I will next proceed to consider the value of contingent reversions. Mr. Jellicoe's formula for the value of a contingent reversion payable on the death of y if x be then alive, is $1-(d_5+\mathrm{P})(1+a_{xy})_{3\frac{1}{2}}$, where P is the office premium for the insurance of £1 on the life of x against that of y, and a_{xy} is to be taken at $3\frac{1}{2}$ per cent interest. This formula is based on the supposition that the amount of the reversion being £1, an insurance of £1 is effected on the life of x against y, the annual premium for which is P; so that, if x die first, £1 is received at the end of the year under the policy; and if y die first, the reversion becomes payable. In this way the reversion is virtually rendered absolute, and is payable on the failure of the joint lives of x and y;

or rather at the end of the year in which that failure takes place. The value of such absolute reversion would be by Mr. Jellicoe's formula

$$1-d_5(1+a_{xy})_{3\frac{1}{2}}.$$

But from this must be deducted the value of the premiums on the insurance, which the purchaser will have to pay. Mr. Jellicoe estimates this at such a sum as an annuity office would require for the grant of an annuity of £P, *i.e.*

$$P(1+a_{xy})_{3\frac{1}{2}}.$$

Thus the value of the contingent reversion becomes

$$1-(d_5+P)(1+a_{xy})_{3\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Assuming the purchaser in this case (or in the case already considered of an absolute reversion) actually to purchase the annuities supposed, his total outlay will be $1-d$, or v. This is easily verified. Thus the annual interest on v is d; the first premium is P, and the cost of an annuity to pay both interest and premium, *i.e.* $(d+P)$, at the end of each year during the joint lives, is $(d+P)a_{xy}$.

Thus, purchase money $=1-(d+P)(1+a_{xy})$
first premium $=P$
cost of annuity of $(d+P)=(d+P)a_{xy}$

and the sum of these is $1-d$, as stated above, or v. In this way, then, the total present cost to a purchaser equals the amount of the reversion, less only a year's discount.

In lieu of the above formula, (3), I propose to substitute the following,

$$1-(d_6+P)(1+a_{xy})_6 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which differs from it, in supposing the discount and the annuity during the joint lives to be both computed at six per cent interest; P having the same meaning as before, viz. the annual office premium for an insurance of £1 on the life of x against y.

Assuming as before the policy of £1 to be effected, and the reversion to be thus rendered absolute, its value, according to the views I have enforced above, will be A_{xy}, or

$$1-d_6(1+a_{xy})_6.$$

The question then arises, what allowance is to be made, or what deduction is to be made from the above value, on account of the cost of the premiums during the joint lives? The deduction made by Mr. Jellicoe's process, *i.e.*, $P(1+a_{xy})_{3\frac{1}{2}}$ would be the proper

one, if an annuity were actually purchased to provide for the premiums as they fall due, or if the premiums were paid up in full at once, or, lastly, if the purchaser found it desirable to keep a sum of money invested in Consols or other readily convertible security, in order to provide for payment of the premiums by means of the interest and sale of portion of the principal from time to time. But the principal purchasers of reversions—the Reversionary and Insurance Companies—do not come under any one of these heads. They would not think of buying an annuity to provide for the premium; nor would they, I imagine, effect the policy by way of single premium. Nor, lastly, is it in the least necessary for them to keep a fund especially appropriated to payment of premiums. On the contrary, they have income coming in from various sources, which they can rely upon with confidence to furnish the funds for payment of the premiums as they fall due. It appears to me therefore that the value of the annuity, or the cost of the premiums, should be calculated at the same rate of interest—say, six per cent—as is used in calculating the value of the reversion. The deduction will be therefore, $P(1+a_{xy})_6$; and we thus get the formula (4) given above. If a sufficient number of contingent reversions be bought at prices found from this formula, they will on the average fall in at such a rate as to return the total outlay in purchase money and premiums, with compound interest at six per cent.

The only valid objection to the above reasoning I am aware of is the one adverted to by Mr. Tucker at the beginning of his paper, vol. v., page 239. The amounts of the reversions purchased being different, some perhaps much larger than the others, it may be said that the principles of average do not properly apply. This appears to me, however, only an argument against purchasing any reversion that shall greatly exceed all the others. It may also, perhaps, be properly admitted as an argument for valuing very large reversions by a less liberal standard than would be applied in other cases. Indeed, the principle is recognized in many instances that any transaction of an unusual character must be charged with a larger margin for contingencies. This is conspicuously the case with "insurances against issue," the premiums for which are invariably calculated with a larger loading than is required in life insurances.

The principles I have above laid down may be applied to find the value of an ordinary policy of insurance, considered as an investment. The sum assured being £1, and the annual premium

P, the value of the policy, supposing the premium to be due and unpaid, will be,

$$(A_x)_6 - P(1+a_x)_6,$$

or

$$1-(d_6+P)(1+a_x)_6 \quad . \quad . \quad . \quad . \quad (5)$$

Having thus cleared the ground, we are now in a position to consider the still more important practical question of the value of a reversionary life interest.

Mr. Jellicoe's formula for the value of a reversionary annuity of £1 for the life of x after the death of y, (which I believe has been very generally, or almost universally, employed,) is

$$\frac{1}{P_x+d_5}-1-(a_{xy})_{3\frac{1}{2}},$$

where P_x is the office premium for the insurance of £1 on the life of x, d_5 is the discount on £1 for a year at 5 per cent, and a_{xy} is the annuity on the joint lives calculated at $3\frac{1}{2}$ (or perhaps $3\frac{1}{4}$ or 3) per cent.

The object of this formula is very easily explained. If the annuity of £1 on the life of x were in possession, its value would be, according to the usual well known formula, $\frac{1}{P_x+d_5}-1$. But the annuity under consideration does not commence until the death of y, if x be then alive, and requires the addition of an annuity of £1 during the joint lives, to convert it into an immediate annuity for the life of x. Now such an annuity could be purchased from a well established Insurance or Annuity Company for $(a_{xy})_{3\frac{1}{2}}$; and deducting this from the value of the annuity in possession, the value of the annuity in reversion is, as given above,

$$\frac{1}{P_x+d_5}-1-(a_{xy})_{3\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

With regard to this formula, it is first to be observed, that the reversionary annuity runs in practice from the death of y; whereas it is virtually supposed in the formula that it runs from the end of the preceding year. For the tabular annuity, a_{xy}, is the value of an annuity which ceases at the end of the year before that in which the joint existence of the two lives x and y fails. By the above formula (6) therefore the purchaser of the reversionary annuity is supposed to receive on the average half a year's annuity in the event of y dying before x, which he will not receive in practice. In strictness, then, there should be subtracted from the above formula $\frac{1}{2}A_{\overset{1}{x}y}$.

It may, perhaps, be considered that in most cases a_{xy} will practically purchase an annuity payable up to the day of the failure of the joint lives (or a *complete annuity*); but this will certainly not be the case when y is very old; and, in that case, the above formula will give too large a value to the reversionary annuity.

I have next to repeat the remark I have made in considering the cases of absolute and contingent reversions, that an Assurance or Reversionary Company purchasing a reversionary annuity would not in practice ever think of purchasing also an annuity during the joint lives; but would take upon itself the risk of loss in consequence of the joint duration of the lives being unusually extended. The question then is, what allowance is to be made for that risk? I would urge that instead of using the two rates of interest $3\frac{1}{2}$ and 5 per cent, the real bearing of the transaction as regards the purchaser, will he better seen, if we employ the same rate throughout. In other words, let the deduction for the value of the annuity during the joint lives be calculated at the same rate of interest as the annuity in possession would yield. In that case, it would not be sufficiently remunerative to assume 5 per cent as the basis of the calculation; and it will be better to take 6 per cent,—the rate used in estimating the values of absolute and contingent reversions. Then the formula becomes, adopting the correction pointed out above,

$$\frac{1}{P_x+d_6}-1-(a_{xy})_6-\tfrac{1}{2}(A_{\underset{xy}{1}})_6 \quad . \quad . \quad . \quad (7)$$

The practical effect of using this formula will be to give a larger value to the reversionary life interest when the reversion is remote, the tenant for life being comparatively young, and to give a smaller value when the tenant for life is very old.

Mr. Scratchley has given (Appendix on Post Obits, p. 18) a formula for the value of a reversionary life interest which becomes when transformed into the notation used in this paper

$$\frac{A_{xy}}{P_x+d}-(1+a_{xy})\frac{P_x}{P_x+d}. \quad . \quad . \quad . \quad . \quad (8)$$

We are told that A_{xy}, a_{xy}, and d are to be "taken at a high rate of interest varying from 6 to 8 per cent"; but these directions are so vague that the formula can scarcely be considered a practical one. The above formula reduces at once to

$$\frac{1}{P_x+d}-1-a_{xy},$$

and Mr. Scratchley has not noticed the necessity of the correction given by the last term of (7).

In using the above formula (7) it is assumed that on the average six per cent interest is realized by the purchaser on the transaction throughout its currency, until the younger life dies; and there can be little doubt, I think, that it is very well worth while for an Insurance or a Reversionary Society to purchase reversionary annuities on such terms. But I am further of opinion that it would be advantageous to an Insurance Office to purchase these reversionary annuities upon terms somewhat higher still. Assuming that such an office would grant loans on life interests at 5 per cent, or would purchase an immediate annuity, perfectly well secured, to pay 5 per cent interest; and would purchase an absolute or contingent reversion to pay six per cent interest; it should seem that the office ought to be satisfied if it purchases reversionary life interests at such prices as will on the average return six per cent interest until the reversions fall in, and five per cent afterwards. I consider it, however, doubtful whether it would be worth while for a Reversionary Society to purchase reversionary life interests on these terms. The advantage of the purchase to an Insurance Company consists not only in the high rate of interest, but in the large insurances which the transaction introduces. If the insurance does not exceed the amount which the office usually retains at its own risk, the office will have the whole of the profit on the insurance; and if the insurance is larger, and has to be distributed among several offices, then the office is indirectly benefited by receiving from those offices other policies in return. For this reason, I consider that an Insurance Company may afford to enter on these transactions upon terms which shew a smaller profit than would satisfy a Reversionary Company.

In order to obtain a formula for the value of a reversionary life interest on the above suppositions, we proceed as follows:—

The value of an annuity of £1 in possession for the life of x is $\frac{1}{P_x+d_5}-1$, and the amount of the insurance to be effected in connection therewith will be $\frac{1}{P_x+d_5}$; the annual premium thereon being $\frac{P_x}{P_x+d_5}$.

Now, supposing this insurance to be effected, the value of the life interest and the insurance together, remains the same what-

ever the number of complete years which may elapse. Whenever therefore the reversionary life interest may fall in, the value of the annuity and the insurance together will *at the end of the year in which y dies* be (the premium on the policy being supposed due and unpaid)

$$\frac{1}{P_x+d_5}-1;$$

or if we assume that a year's annuity is payable at the end of the year in which y dies, the value will be

$$\frac{1}{P_x+d_5}.$$

The *present value* of the annuity and the insurance together at six per cent interest will therefore be

$$\frac{(A_{xy})_6}{P_x+d_5}.$$

But since the reversionary annuity in practice runs only from the death of y, there is on the average only half a year's annuity due at the end of the year in which y dies; and the present value becomes therefore,

$$\frac{(A_{xy})_6}{P_x+d_5}-\tfrac{1}{2}(A_{\underset{xy}{\overline{\ 1}}})_6.$$

Again, since the purchaser has to pay the premiums on the insurance during the joint lives, the present value of such premiums must also be subtracted, which, still using six per cent interest, will be

$$(1+a_{xy})_6\times\frac{P_x}{P_x+d_5}.$$

Thus we get as the consideration to be paid for the reversionary annuity of £1

$$\frac{(A_{xy})_6}{P_x+d_5}-\tfrac{1}{2}(A_{\underset{xy}{\overline{\ 1}}})_6-(1+a_{xy})_6\times\frac{P_x}{P_x+d_5}$$

which becomes, since $A_{xy}=1-d(1+a_{xy})$,

$$\frac{1}{P_x+d_5}-\frac{P_x+d_6}{P_x+d_5}(1+a_{xy})_6-\tfrac{1}{2}(A_{\underset{xy}{\overline{\ 1}}})_6 \quad . \quad . \quad . \quad . \quad (9)$$

Or, since y is usually much older than x, and $A_{\underset{xy}{\overline{\ 1}}}$ is therefore not much less than A_{xy}, we have the simpler formula approximately true

$$\left(\frac{1}{P_x+d_5}-\tfrac{1}{2}\right)(A_{xy})_6-(1+a_{xy})_6\times\frac{P_x}{P_x+d_5}$$

which reduces to

$$\frac{1}{P_x+d_5}-\tfrac{1}{2}-(1+a_{xy})_6\times\left\{\frac{P_x+d_6}{P_x+d_5}-\frac{d_6}{2}\right\}\quad .\ .\ .\ .\ (10)$$

It will be noticed that the substitution of A_{xy} for $A_{\overset{1}{xy}}$ is in favour of the purchaser.

This, then, is the practical formula which I propose should be used in lieu of Mr. Jellicoe's, for calculating the value of reversionary life interests. It will be found to bring out a considerably larger value for such interests when the tenant for life is under 60, but a somewhat smaller value when the tenant for life is more than 65.

I submit that the above formula shows more accurately than the ordinary one the real working of the transaction as it affects the office making an advance. It is true that if an annuity were actually purchased for the joint lives, its cost would have to be calculated at from 3 to $3\frac{1}{2}$ per cent. But, in practice, no such annuity is ever purchased. The office making the advance pays a sum to the borrower, and pays every year the premiums on the insurances effected, *i.e.* virtually makes further advances from year to year. Consistently with what I have said above as to the valuation of absolute and contingent reversions, I hold that the same rate of interest ought to be used in calculating the values of A_{xy} and a_{xy} in the two terms

$$A_{xy}\left(\frac{1}{P_x+d_5}-\tfrac{1}{2}\right),\quad (1+a_{xy})\frac{P_x}{P_x+d_5};$$

and assuming that they are both calculated at six per cent interest, I say that if the original advance and the premiums subsequently paid are accumulated at compound interest at six per cent, then on the average their amount at the time the reversion falls in will be equal to the then value, calculated at five per cent interest, of the annuity and the policies.

The value of the reversionary life interest found by means of the formula (10) being greater than that given by Mr. Jellicoe's formula, when the tenant for life is under 60, the amount of the reversionary annuity and of the redemption money for a given advance will be less; and in the case of remote reversionary interests, the use of the formula (10) will generally give a positive value and enable an advance to be carried out when it would be impracticable according to the ordinary formula. But these, it will be seen, are the cases in which the amount of the insurance bears the largest proportion to the advance; and in these cases an Insurance Company making the advance would directly or in-

directly obtain a large profit on the insurance, and may therefore fairly be satisfied with a rather less profit in other respects.

Whichever of the formulæ is used, the amount of the policy to be effected to secure the return of the capital, and the redemption money, or sum for which the annuity may be redeemed after the reversion has fallen in, will be the same as if the annuity were in possession; *i.e.*, the annuity being £1, the amount of the policy, when either of the formulæ (6) or (10) is used, will be $\frac{1}{P_x+d_5}$ and the redemption money $\frac{v_5}{P_x+d_5}$, or $\frac{1-d_5}{P_x+d_5}$. And when formula (7) is used, the policy will be $\frac{1}{P_x+d_6}$, and the redemption money, $\frac{v_6}{P_x+d_6}$. As Mr. Jellicoe has pointed out, upon the annuity being redeemed, equity requires that the policies effected by the purchaser should be assigned to the vendor; and I believe that the rule of law alluded to by Mr. Jellicoe, that in the absence of any express stipulation, the grantor of the annuity shall not be entitled to the policy, has now been overruled by a decision of Vice-Chancellor Stuart. It is usual to stipulate that upon the redemption of the annuity a proportionate part of it shall be paid from the death of the tenant for life or from the date of the last payment; but here again equity is satisfied if instead of a proportion of the annuity, interest is paid on the redemption money at the agreed rate, five or six per cent.

Instead of insuring the full sum given by the above formula, it will generally be better, and more particularly in the case of remote reversions, to insure *with profits* for a considerably smaller sum, trusting to the reversionary bonuses to increase the insurance in process of time to the required amount. For the bonuses are generally far more than an equivalent for the difference between the participating and non-participating premiums; and if the bonuses are added to the sum assured, we shall virtually have an increasing insurance, amounting at last to a far larger sum than could have been insured for the same premium on the non-participating scale. And as the amount of the advance is increasing year by year, through the payment of premiums and the operation of compound interest, such increasing insurance is better adapted to the circumstances of the case than a uniform insurance would be.

If a reversionary life interest be purchased in its entirety, the policy should be made "whole world"; as otherwise the purchaser

will be exposed to the risk of having to pay a heavy additional premium in the event of the life proceeding to an unhealthy climate. If the extra premium for the whole world licence be an annual one (as is most commonly the case, the charge ranging in ordinary cases from 2*s.* 6*d.* to 10*s.* per £100 assured), P_x will have to be increased in calculating the value of the life interest. If, however, it should be a single premium (say, 10*s.* or £1 per £100 assured) this single payment must be deducted from the value of the life interest, as found from the formula.

But if a portion only of the reversionary life interest be purchased, as in the common case when an advance is made by way of reversionary annuity on security of a reversionary life interest, and there is a considerable margin left, it is not necessary that the policy should be whole world; for in that case it may be stipulated that for every £100 that the purchaser has to pay in extra premiums, the amount of the reversionary annuity shall receive a fixed increase.*

It may be useful to give in conclusion the formulæ for the amounts of the reversionary annuity and the redemption money, in consideration of a present advance of £1; although in practice it will be found more convenient to deduce these from the value of the reversionary annuity as found by the formula (6) (7) or (10). Taking the first of these, we see that a reversionary annuity of £1 is worth $\frac{1}{P_x+d}-1-a_{xy}$, and may be redeemed at any time for $\frac{1-d}{P_x+d}$. Hence a reversionary annuity of $\frac{P_x+d}{1-(1+a_{xy})(P_x+d)}$ is worth £1, and is redeemable for $\frac{1-d}{1-(1+a_{xy})(P_x+d)}$.

Or, the advance being £1

the reversionary annuity $= \frac{P+d_5}{1-(1+a_{xy})_{3\frac{1}{2}}\cdot(P+d_5)}$

the amount of the policy $= \frac{1}{1-(1+a_{xy})_{3\frac{1}{2}}\cdot(P+d_5)}$

and the redemption money $= \frac{1-d_5}{1-(1+a_{xy})_{3\frac{1}{2}}\cdot(P+d_5)}$

where P is the premium for insuring £1 on the life of x.

Next, take the formula (7), but substitute for $A_{\overline{xy}}^{\,1}$, which is troublesome to calculate, A_{xy}, which is nearly equal to it in the common case of y much older than x; then, remembering that $A_{xy}=1-d(1+a_{xy})$, we get the working formula for the value of a reversionary annuity

$$\frac{1}{P+d_6}-\tfrac{1}{2}-\left(1-\frac{d_6}{2}\right)(1+a_{xy})_6 \quad . \quad . \quad . \quad . \quad (11)$$

The advance being, as before, £1, we have now

the reversionary annuity

$$=\frac{P+d_6}{1-\frac{P+d_6}{2}-(1+a_{xy})_6\cdot\left(1-\frac{d_6}{2}\right)(P+d_6)}$$

the amount of the policy

$$=\frac{1}{1-\frac{P+d_6}{2}-(1+a_{xy})_6\cdot\left(1-\frac{d_6}{2}\right)(P+d_6)}$$

and the redemption money

$$=\frac{1-d_6}{1-\frac{P+d_6}{2}-(1+a_{xy})_6\cdot\left(1-\frac{d_6}{2}\right)(P+d_6)}.$$

Lastly, taking the formula (10),

the reversionary annuity

$$=\frac{1}{\frac{1}{P+d_5}-\tfrac{1}{2}-(1+a_{xy})_6\cdot\left(\frac{P+d_6}{P+d_5}-\frac{d_6}{2}\right)}$$

$$=\frac{P+d_5}{1-\frac{P+d_5}{2}-(1+a_{xy})_6\cdot\left\{P+d_6-\frac{d_6}{2}(P+d_5)\right\}}$$

the amount of the policy

$$=\frac{1}{1-\frac{P+d_5}{2}-(1+a_{xy})_6\cdot\left\{P+d_6-\frac{d_6}{2}(P+d_5)\right\}}$$

and the redemption money

$$=\frac{1-d_5}{1-\frac{P+d_5}{2}-(1+a_{xy})_6\cdot\left\{P+d_6-\frac{d_6}{2}(P+d_5)\right\}}.$$

By the help of these formulæ, it is not difficult to prove that the amount of the policy is always greater when formula (11) is used, than that given by the use of (10). But that the redemption money is not always greater. In fact, the redemption money obtained from (11) is greater or less than that obtained from (10) according as $1+a_{xy}$ is greater or less than $\dfrac{1}{\frac{2P}{1-P}+d_6}$. The values however of the redemption money as found from these two formulæ are very nearly equal, as will be seen by inspection of the following tables.

TABLE I.—*Immediate Annuities, 5 and 6 per Cent.*

Table showing the amounts of the policy and of the redemption money when an annuity of £1 is purchased,—allowing for insurance at the average annual premiums here given, and returning the purchaser 5 and 6 per Cent respectively on his outlay.

Age.	Premium.	5 PER CENT.		6 PER CENT.	
		Policy.	Redemption Money.	Policy.	Redemption Money.
x	P_x	$\frac{1}{P_x+d_5}$	$\frac{1-d_5}{P_x+d_5}$	$\frac{1}{P_x+d_6}$	$\frac{1-d_6}{P_x+d_6}$
	£ s. d.				
20	1 15 1	15·347	14·616	13·488	12·724
25	1 19 6	14·843	14·136	13·098	12·356
30	2 4 9	14·286	13·606	12·661	11·945
35	2 11 2	13·661	13·010	12·168	11·479
40	2 19 2	12·953	12·336	11·604	10·947
45	3 5 7	12·134	11·556	10·942	10·323
50	4 3 6	11·189	10·656	10·168	9·592
55	5 2 0	10·140	9·657	9·294	8·768
60	6 7 8	8·973	8·546	8·304	7·834

TABLE II.—*Values of Reversionary Annuities.*

(*a*)

As found by Mr. Jellicoe's formula No. (6), $\frac{1}{P+d_5}-(1+a_{xy})_{3\frac{1}{2}}$.

Younger Age. x.	Difference of age $=y-x$.								
	10	15	20	25	30	35	40	45	50
20	−1·308	−·689	·098	·907	1·990	3·422	4·946	6·221	7·753
25	− ·864	−·129	·633	1·672	3·061	4·549	5·795	7·301	8·709
30	− ·350	·354	1·338	2·677	4·123	5·338	6·814	8·201	9·204
35	·014	·929	2·208	3·610	4·792	6·241	7·608	8·598	
40	·549	1·742	3·075	4·211	5·622	6·962	7·935		
45	1·156	2·404	3·480	4·849	6·167	7·130			
50	1·767	2·739	4·023	5·286	6·216				
55	2·137	3·278	4·436	5·297					
60	2·547	3·574	4·347						

(*b*)

As found by formula (11), $\frac{1}{P+d_6}-\frac{1}{2}-\left(1-\frac{d_6}{2}\right)(1+a_{xy})_6$.

Younger Age. x.	Difference of age $=y-x$.								
	10	15	20	25	30	35	40	45	50
20	·700	1·005	1·422	1·840	2·462	3·377	4·416	5·294	6·433
25	·803	1·197	1·595	2·193	3·085	4·101	4·961	6·082	7·190
30	·961	1·334	1·906	2·769	3·761	4·600	5·701	6·793	7·597
35	1·010	1·549	2·380	3·345	4·166	5·246	6·326	7·119	
40	1·196	1·981	2·907	3·698	4·753	5·812	6·593		
45	1·461	2·339	3·094	4·124	5·169	5·942			
50	1·771	2·461	3·435	4·440	5·191				
55	1·921	2·795	3·725	4·422					
60	2·153	2·985	3·621						

TABLE II.—(continued).

(*c*)

As found by the new formula No. (10), $\frac{1}{P+d_5} - \frac{1}{2} - (1+a_{xy})_6 \times \left(\frac{P+d_6}{P+d_5} - \frac{d_6}{2}\right)$

Younger Ages. x	Difference of age $= y - x$. 10	15	20	25	30	35	40	45	50
20	·816	1·165	1·640	2·118	2·828	3·873	5·059	6·062	7·362
25	·930	1·378	1·831	2·511	3·525	4·681	5·659	6·933	8·194
30	1·107	1·530	2·177	3·153	4·276	5·227	6·473	7·710	8·619
35	1·157	1·764	2·700	3·787	4·711	5·928	7·144	8·038	
40	1·359	2·238	3·275	4·160	5·342	6·528	7·402		
45	1·646	2·623	3·462	4·608	5·770	6·630			
50	1·976	2·737	3·811	4·921	5·750				
55	2·123	3·078	4·095	4·858					
60	2·353	3·255	3·938						

TABLE III.—*Annuity which £1 will purchase, and its Redemption Money.*

Ages. x	y	J, (6). Annuity.	J, (6). Redemption Money.	6 PER CENT, (11). Annuity.	6 PER CENT, (11). Redemption Money.	S, (10). Annuity.	S, (10). Redemption Money.
				Difference of age, 10 years.			
20	30	..	..	1·4286	18·178	1·2255	17·912
25	35	..	..	1·2453	15·387	1·0753	15·200
30	40	..	..	1·0406	12·429	·9033	12·290
35	45	71·4286	929·286	·9901	11·366	·8643	11·245
40	50	1·8215	22·470	·8361	9·153	·7358	9·077
45	55	·8651	9·997	·6845	7·066	·6075	7·020
50	60	·5659	6·030	·5647	5·416	·5061	5·393
55	65	·4679	4·519	·5206	4·564	·4710	4·549
60	70	·3926	3·355	·4645	3·639	·4250	3·632
				Difference of age, 15 years.			
20	35	..	..	·9950	12·661	·8584	12·546
25	40	..	..	·8354	10·322	·7257	10·259
30	45	2·8249	38·435	·7496	8·954	·6536	8·893
35	50	1·0764	14·004	·6456	7·411	·5669	7·376
40	55	·5741	7·082	·5048	5·526	·4468	5·512
45	60	·4160	4·807	·4275	4·413	·3812	4·405
50	65	·3651	3·891	·4063	3·897	·3654	3·894
55	70	·3051	2·946	·3578	3·137	·3249	3·138
60	75	·2798	2·391	·3350	2·624	·3072	2·625
				Difference of age, 20 years.			
20	40	10·2041	149·142	·7032	8·948	·6098	8·913
25	45	1·5798	22·331	·6270	7·747	·5461	7·720
30	50	·7474	10·169	·5247	6·267	·4593	6·249
35	55	·4529	5·892	·4202	4·824	·3704	4·819
40	60	·3252	4·012	·3440	3·766	·3053	3·766
45	65	·2874	3·321	·3232	3·336	·2889	3·339
50	70	·2486	2·649	·2911	2·792	·2624	2·796
55	75	·2254	2·177	·2685	2·354	·2442	2·358
60	80	·2300	1·966	·2762	2·164	·2539	2·170

TABLE III. (continued).

Ages.		J, (6).		6 PER CENT, (11).		S, (10).	
x	*y*	Annuity.	Redemption Money.	Annuity.	Redemption Money.	Annuity.	Redemption Money.
Difference of age, 25 years.							
20	45	1·1025	16·114	·5435	6·916	·4721	6·900
25	50	·5981	8·455	·4560	5·634	·3982	5·629
30	55	·3736	5·083	·3611	4·313	·3172	4·316
35	60	·2770	3·604	·2990	3·432	·2641	3·436
40	65	·2375	2·930	·2704	2·960	·2404	2·966
45	70	·2062	2·383	·2425	2·503	·2170	2·508
50	75	·1892	2·016	·2252	2·160	·2032	2·165
55	80	·1888	1·823	·2261	1·982	·2058	1·988
Difference of age, 30 years.							
20	50	·5025	7·345	·4062	5·169	·3536	5·168
25	55	·3267	4·618	·3241	4·005	·2837	4·010
30	60	·2425	3·299	·2659	3·176	·2339	3·182
35	65	·2087	2·715	·2400	2·755	·2123	2·762
40	70	·1779	2·195	·2104	2·303	·1872	2·309
45	75	·1622	1·874	·1935	1·997	·1733	2·003
50	80	·1609	1·715	·1926	1·848	·1739	1·853
Difference of age, 35 years.							
20	55	·2922	4·271	·2961	3·768	·2582	3·774
25	60	·2198	3·107	·2438	3·012	·2136	3·020
30	65	·1873	2·548	·2174	2·597	·1913	2·603
35	70	·1602	2·084	·1906	2·188	·1687	2·195
40	75	·1436	1·732	·1721	1·884	·1532	1·890
45	80	·1403	1·621	·1683	1·737	·1508	1·743
Difference of age, 40 years.							
20	60	·2022	2·955	·2264	2·881	·1977	2·890
25	65	·1726	2·440	·2016	2·491	·1767	2·498
30	70	·1468	1·997	·1754	2·095	·1545	2·102
35	75	·1314	1·710	·1581	1·815	·1400	1·822
40	80	·1260	1·554	·1517	1·661	·1351	1·667
Difference of age, 45 years.							
20	65	·1607	2·349	·1889	2·404	·1650	2·412
25	70	·1370	1·937	·1644	2·031	·1442	2·039
30	75	·1219	1·659	·1472	1·758	·1297	1·765
35	80	·1163	1·513	·1405	1·613	·1244	1·619
Difference of age, 50 years.							
20	70	·1290	1·886	·1554	1·977	·1358	1·985
25	75	·1148	1·623	·1391	1·719	·1220	1·725
30	80	·1086	1·478	·1316	1·572	·1160	1·578

These tables have been calculated by Mr. Berridge, of the London and Provincial Law Assurance Society, and are arranged so as to show side by side the results of the principal formulæ mentioned in this paper. The premiums involved are the average of the non-participating premiums charged by six leading offices; they are given in Table I. This Table also shows the sum to be assured to cover the risk incident on the purchase of an annuity of £1, whether immediate or not; and the redemption money for the same; being the values of the expressions $\frac{1}{P_x+d}$ and $\frac{1-d}{P_x+d}$. They are calculated at 5 and 6 per cent.

The second table gives the value of a reversionary annuity of £1 according to three different methods of valuation. The first division of the table marked (*a*) results from the employment of Mr. Jellicoe's formula $\frac{1}{P_x+d_5}-1-(a_{xy})_{3\frac{1}{2}}$, the Carlisle 3½ per cent annuity plus unity being subtracted from the values in the column of Table I. headed $\frac{1}{P_x+d_5}$.

In calculating the second division (*b*), formula (7) was used; the joint life single premium A_{xy} being substituted for the contingent premium $A_{\overset{1}{x}y}$. Interest at 6 per cent is therefore reserved throughout. Identical results would of course have been given by formula (11).

The third division (*c*) contains the values of a reversionary annuity of £1 according to my new formula (10), the Carlisle 6 per cent annuity being used in this and the previous division.

The third table gives the reversionary annuity which £1 will purchase and its redemption money, according to the same three methods, the values in the annuity columns being the reciprocals of those in Table II. The amounts of redemption money opposite to them result from their multiplication by the corresponding amounts in Table I., the five per cent values being employed for the columns headed J and S, and the six per cent values for the middle column; or rather the logarithms of which the first table contains the natural numbers. Where blanks occur the formula gives a negative value to the reversionary annuity.

The following summary of the discussion which followed the reading of the paper is abridged from the *Insurance Record*.

The PRESIDENT, in inviting discussion, while regretting the absence of Messrs. Jellicoe and Tucker, hoped that those gentlemen who were

connected with Reversionary Societies would favour the meeting with their views. He thought that Mr. Sprague's proposed method of dealing with reversionary securities was one which, if acted on, would drive both the public and the Reversionary Societies out of the market.

Mr. BUNYON thought that Mr. Sprague was taking too sanguine a view of the purchase of reversions. Looking at Mr. Jellicoe's formula, he did not see what provision was made for "profit" on the transaction. If the profit was supposed to come from the rate of interest secured (say 5 per cent), it amounted to nothing more than a rate of interest, which they were able to obtain from certain other securities. If from the premium charged, a considerable portion did not reach them, by reason of their having to distribute the assurances, often in the vain hope of some day getting a return. If from the value of the annuity reserved at $3\frac{1}{2}$ per cent, he was still more doubtful, and was quite willing to let all annuity business on those terms go elsewhere. He was aware that considerable profit had been made upon reversionary transactions, and that one reason for granting better terms might be the fact that there was no selection against the purchaser. He thought, however, that as a rule there was no great value in selection, and concluded by entreating purchasers not to spoil the market.

Mr. H. AMBROSE SMITH preferred as a matter of account to abide by the principle typified in Mr. Jellicoe's formula, and in buying a reversion would also suppose the annuity bought, and set up the transaction in both the Investment Ledger and Life Contingency accounts. He thought Mr. Sprague's formula inapplicable where only a small number of contingencies were at stake.

Mr. HODGE felt inclined to agree with the mathematical views taken by Mr. Sprague, though in some points not so entirely. The market for reversionary securities was a limited one, and the competition of Assurance Companies would have the effect of raising the price of reversions, and might render it impossible for the Reversionary Companies to continue their business; a result not greatly to be lamented, as they would be enabled to dispose of their property at a considerable profit. He considered that the expectations of profit from these transactions were exaggerated. Of the seven Reversionary Companies which had started, three had ceased to exist; and of those surviving, none had their shares at a premium. The purchase of a contingent reversion, according to the ordinary formulæ laid down in the text books, cannot be said to be an investment, but a speculation. Such a transaction amounts to doing the business of an insurance office, without getting the profit. He believed that, in the cases which generally arose in practice, Mr. Jellicoe's formulæ for the valuation of contingent reversions would be inapplicable, and that even Mr. Sprague's would often bring out results so small that Reversionary Companies would decline to enter into the business. The latter formula [*i.e.* No. (4) in the paper] had been known to many actuaries and commonly used for many years. He was not aware of any case in which a Reversionary Company had acted upon Mr. Jellicoe's formula; but it is one properly adapted for private purchasers, who can thus obtain a certain result without taking upon themselves the chance of either great loss or great gain. The method usually adopted by purchasers of covering themselves by assuring the life of the reversioner against that of the life-tenant is convenient and advantageous to the vendor, but it is not so

strictly legal as the insurance of a sum payable upon the decease of the life-tenant, contingent upon his dying second. He thought that all the formulæ, though convenient, did not quite meet the actual conditions of practice, for annuities were generally payable either half-yearly or quarterly, and the life-tenant of an estate was entitled to an apportionment of the income up to the date of his death. He should be glad to hear of some method for approximately arriving at the value of contingent reversions, including the benefit of survivorship.

Mr. BAILEY congratulated the Institute that a paper of so practical a character was presented on the opening night of the session. He did not agree with Mr. Sprague's reasons for adopting the rates of interest employed in the paper. He thought that at present reversions should be valued at 6 per cent, because in fact that was the price which at present ruled in the market; but, for the future, owing to the protection afforded by the recent legislation, their value would no doubt be enhanced, and the rate of interest consequently obtainable be lessened. Assurance Companies might with advantage purchase at a lower rate than 6 per cent, because they cannot realize such a return upon their ordinary investments. The principle originally proposed by Mr. Griffith Davies for valuing an isolated annuity was extended by Mr. Jellicoe to the case of a single reversion. He (Mr. Bailey) did not, however, think that it had been much adopted in practice; for, the capital of the purchaser being secure, his sole risk was as to the rate of interest which he would ultimately make, and he was generally willing to take that speculation on himself. He would remark that the formulæ generally given did not exactly represent the conditions of real life. The annuity actually purchaseable, being one payable up to the day of death, was not that of the text books; and the assurance did not become payable at the end of the year, but usually 3 months after proof of the decease of the reversioner. He thought that the amount of assurance required by all the formulæ in connection with reversionary annuities was excessive, and that larger amounts might be raised and the transaction be secured by effecting such an increasing assurance (to increase by a fixed sum annually for, say, 20 years) as would provide for the outlay of premiums and interest, charging the whole by way of ordinary mortgage. He knew that offices had an objection to granting assurances increasing without limit. As showing the expediency of further discussing the subject, he instanced a case which was submitted several years ago to five well-known Actuaries, where the redemption money for the contingent annuity was fixed by A at £282,000, by B at £237,000, by C at £184,000, by D at £154,000, and by E at £47,000. He did not see much in the modifications proposed by Mr. Sprague, since the practical differences between his formulæ and Mr. Jellicoe's were not important; and they did not obviate the necessity of saddling the vendor with the cost of the enormous amount of assurance required in the early years of the transaction.

Mr. COLES, quoting the amounts of the capitals of the existing Reversionary Companies, stated that the total was little more than £2,000,000; and he thought that the amount distributed amongst the Insurance Offices would scarcely be felt. He wished to know whether the reversions offered ran in large amounts, or were of sufficiently uniform amounts to be taken up by Companies of ordinary capital.

Mr. BADEN understood Mr. Sprague to treat the question as one, not of

profits, but of investment merely. The problem was to fix the rate of interest, so as to secure a return of the capital improved at that rate. He should take the annuity upon a true table of mortality, at 4 per cent interest, if the office generally improved its funds at that rate; and d at 5 per cent. The result would nearly correspond to taking both the annuity and d at 6 per cent, and we should have a better *rationale*.

The PRESIDENT anticipated an extension in this class of business, by reason of the increase in marriage settlements and the improvement in the value of property. Therefore, he did not think that they would be necessarily driving the Reversionary Companies out of the field; at all events, for some considerable time. It was very advisable to see by the mathematical formulæ what terms can be safely given, even if they were higher than those hitherto prevailing. He thought a Company need not actually purchase the annuity, but lay out its money and trust to averages. A higher rate of interest must be allowed for in cases of long deferred reversions, since they were likely to be kept out for a considerable period of the interest upon a heavy investment of capital.

Mr. SPRAGUE, in reply, was glad to have had so full a meeting and that so many gentlemen of practical experience had come forward. By thus interchanging ideas, greater progress was secured than could possibly be by separate study. His experience differed from Mr. Bunyon's in the matter of a return for those assurances given away; as he found that he always sooner or later received a fair equivalent. He certainly agreed with Mr. Bunyon that it was undesirable to spoil the market. He did not propose that there should be no deviation from the rates he named, but desired principally to show how to calculate the real working of the transaction. He contended that the same rate of interest should be maintained throughout the formulæ, since the considerations attending the grant of an annuity to the public differed from those involved in these transactions. The reserved value of the annuity was in fact employed in the transaction itself, and in the others of the same nature, and realized the same rate of interest. He thought the effect of "selection" was traceable for many years, and he therefore objected to the grant of increasing assurances; for without doubt every year the life was, on the average, getting worse. He was aware of the technical imperfection in the formula for the annuity value, as pointed out by Mr. Hodge. But, as the assurance premiums were generally payable yearly, the ordinary correction to add $\frac{1}{4}$ for half yearly payments was a close approximation. To this point he hoped one day to return, if not anticipated. He was afraid that the amounts of reversions offered for sale did not run uniformly. However that was in his opinion no serious objection to Insurance Companies dealing with them. He thought that an office might safely invest 20 per cent of its funds in these securities—for then even the very worst experience would only reduce the average rate of interest realized on the whole assets from, say, 5 per cent to 4 per cent.

On the determination of the Average Risk attaching to the grant of Insurances upon Lives. By DR. M. KANNER, *Actuary of the Frankfort Life Assurance Company.*

[Translated from the *Deutsche Versicherungs-Zeitung.*]*

AN Insurance Office, which in consideration of fixed premiums undertakes the payment of a sum on the death of the assured, strictly speaking lays a wager with each of the assured, in which the stakes are proportional to the probabilities of the happening of the two contrary events—his living and his dying. Every assurance of a sum payable at death admits of being divided into partial assurances for a fixed period, as a year. Let the sum assured be S and the probability of death within a year p, then pS will be the stake of the assured, or the premium; and $(1-p)$S the stake of the office, *i.e.* the amount which the office would lose in case of the death of the assured within the year. The two stakes together therefore amount to the sum assured.

If the number of the assured be sufficiently large, it may happen that the office on the whole neither gains nor loses. Of all the cases that may happen, this is always the most probable, if all the lives are insured for the same amount and have the same probability of death. Moreover, the probability of this case increases with the number of the assured, so that it approaches without limit to certainty, and would reach certainty if the number of the assured could become infinitely great. This proposition is based on the Principle of Large Numbers, of which the celebrated mathematician, James Bernouilli, succeeded in obtaining a strict demonstration after twenty years' reflection; and which then ceased to be a mere result of observation, and was acknowledged as a mathematical truth.

The case in which the office neither gains nor loses is only physically possible in the rare event when, the sums assured being all equal, and n in number, pn is an integer; or when, the sums assured being different, some combination of the deaths is possible, such that the sum of the claims to be paid is equal to the sum of the premiums. The greater the number of the assured, and the more nearly the sums assured approach equality, the greater also will be the probability of there being very little gain or loss; and therefore it is evident that the risk of a deviation from the most probable case becomes greater as the numbers are smaller, and the sums assured more unequal.

* We are indebted to Mr. George Humphreys and to Mr. J. Hill Williams for much assistance in the preparation of this translation.

Insurance Companies calculate their rates on the supposition that the most probable case will happen; so that their profit must consist only of the excess of interest realized beyond that calculated upon, and of a portion of the loading, or addition to the net premium. So also, the Premium-Reserve is formed by the accumulation at interest of the balance of the premiums which would remain at the time of valuation after payment of the claims, if the most probable case had constantly happened. This Reserve again has this further characteristic, that on the supposition of the most probable case happening for the future, it will exactly suffice for the discharge of the liabilities undertaken by the office.

From this it results that the Premium-Reserve by itself does not offer the guarantee that may reasonably be desired; since the possibility of a loss is not provided for. For this purpose, therefore, every Insurance Office ought, year by year, to have a separate Fund (Capital-Reserve), the magnitude of which is to be determined at the beginning of each year according to the *amount of the insurances in force.** To determine what this Fund should be, is a problem in probabilities; for we cannot think of providing for all possible risks to their full value, but we may calculate the *average risk*, by taking into account every possible case according to its probability.

From the foregoing explanation of the nature of the risk, it is evident that this is still the case if the fundamental Mortality Table is perfectly accurate, *i.e.*, if it represents the actual probabilities of death. But absolute accuracy is never to be obtained, and this circumstance is a fresh danger for the Insurance Office, which however is met by the addition made to the net premiums.

It is not my present purpose to discuss the theory of the construction of Tables of Mortality, and I will now only point out that a Table, founded upon the arithmetic mean of a series of observations free from error, made independently of each other, and under similar conditions, indicates the most probable values of the actual probabilities of death, whilst these last must necessarily remain unknown.

Suppose that we have for a given age the following data:—

Out of n_1 living, m_1 have died within a year

" n_2 " m_2 " "

$\vdots \qquad \vdots$

" n_k " m_k " "

* The word in the original, here and elsewhere, is *Versicherungsbestand*, which the author states would be expressed in French by *état des assurances*.

Then the greatest probability is to be ascribed to that value of the unknown probability of death, by the adoption of which the concurrence of the observed results has the greatest probability.

Let x be any probability of death between 0 and 1, and for brevity put

$$n_1+n_2+\ldots+n_k=\Sigma(n)$$
$$m_1+m_2+\ldots+m_k=\Sigma(m)$$

Then the probability of any one of the above results $(m.n)$, (that of n living, m die within a year) will be expressed by

$$\mathrm{B}.x^m(1-x)^{n-m}$$

where B denotes the mth coefficient of the expansion of $(1+x)^n$ $\left[=\frac{\lfloor n}{\lfloor m \lfloor n-m}\right]$. The probability of the happening of all the k independent events is equal to the product of their separate probabilities, *i.e.* to

$$\mathrm{C}.x^{\Sigma(m)}(1-x)^{\Sigma(n)-\Sigma(m)}$$

where C denotes the product of all the k Binomial coefficients. This product is to be a maximum with regard to x. Hence putting the first differential coefficient equal to 0, we obtain, (omitting the constant factor C)

$$\Sigma(m)x^{\Sigma(m)-1}(1-x)^{\Sigma(n)-\Sigma(m)}-\{\Sigma(n)-\Sigma(m)\}x^{\Sigma(m)}(1-x)^{\Sigma(n)-\Sigma(m)-1}=0$$

whence $$x=\frac{\Sigma(m)}{\Sigma(n)}.$$

We can easily prove that for this value, the second differential coefficient is negative, and therefore a maximum occurs; and thus is established the proposition, that the arithmetical mean represents the most probable value of the probability of death.

From the foregoing it evidently follows that there always exists a risk for the office independently of the unavoidable errors in the construction of the Table of Mortality.

Now in order to determine the risk for a given amount of existing insurances, and for the period of one year, I start from the simple principle, that the premium which the assuring office would have to pay to another, in order to insure itself against every possible loss during the year, furnishes a measure of the risk, or represents *the average risk,* an expression which will be justified further on.

If we wished to calculate this premium directly, we must take into consideration, all the 2^n possible cases which may arise among n assured persons, and multiply the loss in each case into the

probability of its occurrence; and the sum of all these products would represent the premium sought. But we see that such a process would be extremely laborious, and that with only a moderate number of lives insured many generations would be required for performing the calculations. But there fortunately exists another method by which we can arrive at the result without trouble. This is by calculating, not the mathematical expectation of *loss*, but that of *gain*, these two being, as will be shown directly, always equal to each other.

The loss in consequence of the death of a person of the age x insured for S, would consist of S diminished by the last reserve, R_x, and the last premium, ϖ_x, and a year's interest upon both of these, *i.e.*

$$S-(R_x+\varpi_x)(1+i).$$

But there exists between the old Reserve, R_x, and the new, R_{x+1}, the following easily proved relation

$$(R_x+\varpi_x)(1+i)=R_{x+1}+p(S-R_{x+1}),$$

where p denotes the probability of dying in the $(x+1)$th year. We can therefore bring the actual loss into the following form,

$$S-R_{x+1}-p(S-R_{x+1}).$$

If now we call $S-R_{x+1}$ the *reduced* sum assured, and $p(S-R_{x+1})$ the *reduced* premium, we can enunciate in the following terms the theorem mentioned above. *If there are n persons of the same age, whose probability of dying in a year is p, and whose reduced sums assured are $a_1, a_2, \ldots a_n$, the sum of the mathematical expectations of gain is equal to the sum of the mathematical expectations of loss.*

Proof.—The probability that m specified persons insured for the reduced sums $a_p, a_q, a_r, \ldots$ will die, and the remaining, $n-m$, will live, is represented generally by $p^m(1-p)^{n-m}$. If we consider loss as negative gain, we can represent the gain or loss, whichever it may be, in this case by

$$p\Sigma(a)-(a_p+a_q+\ldots\ldots+a_t) \quad . \quad . \quad . \quad (1)$$

where for brevity, we put

$$a_1+a_2+\ \ldots\ +a_n=\Sigma(a)$$

and where $p\Sigma(a)$ represents nothing else than the expected loss of the office. The mathematical expectation of gain or loss for the case supposed is therefore generally

$$p^m(1-p)^{n-m}\{p\Sigma(a)-(a_p+a_q+\ \ldots\ +a_t)\} \quad . \quad . \quad (2)$$

and we shall get the several mathematical expectations for all possible cases, by giving m in the above expression all integral values from 0 to n in succession, and then for each value of m forming all the combinations of the quantities $a_1, a_2, a_3, \ldots\ldots a_n$, taken m at a time. The number of all the cases is equal to the sum of all the coefficients of $(1+x)^n$, that is, to 2^n.

In order to find the value of all the gains or losses that happen for a given value of m, we have to find the value of the expression (1) for each of the possible $\dfrac{n(n-1)(n-2)\ldots}{\underline{|m}}$ cases, and to take the sum of these values. We get as the positive part of that sum.

$$\frac{n(n-1)\ldots\ldots}{\underline{|m}}p\Sigma(a).$$

The negative part consists of the sum of the several terms of all the combinations, m at a time, of the quantities $a_1, a_2, \ldots a_n$. In that sum, every quantity must occur $\dfrac{(n-1)(n-2)\ldots}{\underline{|m-1}}$ times; for if we consider those combinations in which a_p occurs, we see that they are so formed that a_p occurs along with all the possible combinations, $(m-1)$ at a time, of the remaining $(n-1)$ quantities, so that the number of the combinations in which a_p occurs, is

$$\frac{(n-1)(n-2)\ldots}{\underline{|m-1}}.$$

Since this is true for each of the m quantities, we obtain as the negative part of the sum,

$$\frac{(n-1)(n-2)\ldots}{\underline{|m-1}}(a_1+a_2+\ \ldots\ +a_n)=\frac{(n-1)(n-2)\ldots}{\underline{|m-1}}\Sigma(a).$$

Combining this with the positive part as found above, we get the sum equal to

$$\Sigma(a)\left\{\frac{n(n-1)\ldots}{\underline{|m}}p-\frac{(n-1)(n-2)\ldots}{\underline{|m-1}}\right\}.$$

This expression may also be written

$$\frac{\Sigma(a)}{n}\left\{\frac{n(n-1)\ldots}{\underline{|m}}np-\frac{n(n-1)\ldots}{\underline{|m-1}}\right\}=\frac{\Sigma(a)}{n}\cdot\frac{n(n-1)\ldots}{\underline{|m}}(np-m)$$

The sum of all the mathematical expectations of gain or loss for a given value of m is therefore

$$\frac{\Sigma(a)}{n}\cdot\frac{n(n-1)\ldots}{\underline{|m}}(np-m)p^m(1-p)^{n-m}.$$

Giving m now every value from 0 to n, we finally obtain as the sum of the mathematical expectations for all the 2^n cases,

$$\frac{\Sigma(a)}{n}\left[np(1-p)^n+(np-1)n(1-p)^{n-1}+\ldots+(np-n)p^n\right]$$

The expression in the square brackets vanishes identically; for we can also write it in the following form

$$np\{(1-p)^n+n(1-p)^{n-1}p+\ldots+p^n\}$$
$$-np\{(1-p)^{n-1}+(n-1)(1-p)^{n-2}p+\ldots+p^{n-1}\}$$

and it is evident, that

$$(1-p)^n+n(1-p)^{n-1}p+\ldots+p^n=\{(1-p)+p\}^n=1$$

and $(1-p)^{n-1}+(n-1)(1-p)^{n-2}p+\ldots+p^{n-1}=\{(1-p)+p\}^{n-1}=1$

so that the expression in the square brackets reduces to $np-np=0$. But if the whole expression vanishes, the sum of the positive terms must be equal to the sum of the negative; *i.e.* the sums of the mathematical expectations of gain and loss are equal to each other.

We might give the proposition just proved the greatest generality of which it admits, and prove that

For any amount of insurances, whatever may be the ages of the lives insured, and whatever the nature of the insurances, the sums of the mathematical expectations of gain and loss for any interval of time, are equal to each other, if only the premiums are calculated upon the supposition of the most probable case.

Though we do not here require that greater generality, yet I venture to think that in the foregoing demonstration I have pointed out the way to generalization.

It may now be useful to illustrate the method of calculation by two examples.

1. *Problem.*—Let it be required to determine the average risk during a year, when 15 persons of the same age are insured, each for the reduced sum of 5000 thalers, the probability of death within a year being ·02.

Solution.—The sum of the reduced premiums $p\Sigma(a)$, which occurs in formula (2) is here equal to $\cdot 02\times 15\times 5000=1500$, which amount the office would expect to lose in the most probable case. Whatever beyond this amount it might pay, would be a *loss* in the sense in which we have throughout wished the term to be understood.

There are here $2^{15}-1$ cases of loss; because there is only one case in which the office gains, that, namely, in which no

death occurs. The mathematical expectation of gain is therefore $\cdot98^{15} \times 1500 = 1108$, which value represents at the same time the average risk.

2. *Problem.*—Let the following insurances be given with the common probability of death ·03.

	No. of Persons insured.		Reduced Sums assured.	Reduced Premiums.
A	1	insured for 5000 each	5000 Thlr.	150 Thlr.
B	6	„ 3000 „	18000 „	540 „
C	10	„ 1000 „	10000 „	300 „
D	30	„ 400 „	12000 „	360 „
Total	47		45,000 Thlr.	1,350 Thlr.

What is the average risk within a year?

Solution.—The cases of gain are as follows:

(1)	There may die 0 person	——	in which case there is a gain	1350
(2)	„ 1 „	out of C	„ „ „	350
(3)	„ 1 „	out of D	„ „ „	950
(4)	„ 2 persons	out of D	„ „ „	550
(5)	„ 3 „	out of D	„ „ „	150

If these cases are separately analyzed and their probabilities determined, we shall find as the mathematical expectation of gain,

$$97^{47} \times 1350 = 322{\cdot}56$$
$$10 \times \cdot03 \times \cdot97^{46} \times 350 = 25{\cdot}86$$
$$30 \times \cdot03 \times \cdot97^{46} \times 950 = 210{\cdot}60$$
$$\frac{30.29}{2} \times \cdot03^2 \times \cdot97^{45} \times 550 = 54{\cdot}68$$
$$\frac{30.29.28}{2\cdot3} \times \cdot03^3 \times \cdot97^{44} \times 150 = 4{\cdot}30$$
$$\text{Total} \quad . \quad . \quad 618{\cdot}00$$

The average risk is therefore $=618$.

If we should have to do with a large number of persons insured for very different sums, the calculations lead, as in the most important applications of the theory of probabilities, to products of a very great number of unequal factors, in which cases we must have recourse to the method of approximation given by Laplace in his great work—*Théorie analytique des probabilités*—which requires for its demonstration the most refined analysis.

The above given method of calculating the average risk extends to all kinds of Life Insurances, as for example, to joint lives and the like. But for Endowments, in which on the contrary the number of the cases of loss is far smaller, we can determine by a direct process the sum of the mathematical expectations of loss.

I now turn to the practical side of the subject, in order to place the application of the mathematical idea of the average risk in a clear light. The opinions of authors in regard to this idea are as wide apart from one another as their theories and the results to which they are led. Is then the question of Risk really of such a vague nature as the various methods of treating it would appear to indicate? From a practical point of view this is undoubtedly the case; but by no means from the mathematical point of view, provided that the question itself is put mathematically.

Thus as long as we simply speak of *a risk*, we have just as little in view a mathematically defined idea as when we speak of the further *duration of life* by itself; but just as this last is only determined by means of its average value, so also we cannot speak of Risk in a mathematical sense, until we introduce its average value. The Risk to an Insurance Office is every disadvantageous possibility of a loss of any amount, and is therefore in itself indeterminate. But if we imagine any amount of existing insurances frequently repeated, so perhaps, that we represent to ourselves an indefinite number of offices all with the same amount of insurances in force, then will the total result in a given time for all the offices taken together, show neither gain nor loss, if their number is assumed infinitely great. The individual offices nevertheless will have to show, some, gains, and others, losses, of various magnitudes; and indeed, all possible cases will appear in proportion to their respective probabilities. The limit of the ratio of the total losses to the number of offices represents the *average* loss, which is equal to the *average* gain; and herein lies the practical meaning of the *average risk*, as well as the justification of the term.

There arises now the important question:—Is this average risk proper to take the place of the Capital-Reserve? Or, in other words, is the office perfectly secure if it possesses such a Fund in addition to its Premium-Reserve?

This question withdraws itself from mathematical treatment because here the question is as to a magnitude which can only be arbitrarily assigned, and for the determination of which, mathematically expressed conditions are not attainable. The question here is regarding a number dependent on opinion, by means of which it is to be specified, from what multiple of the average risk the Capital-Reserve must be formed. This number would determine the degree of security of the office, whilst the average risk itself represents the measure of the danger for different amounts of existing insurances.

The determination of the average risk is of particularly great importance for Mutual Companies. Suppose, for instance, that there are two Mutual Insurance Offices, the one with the amount of insurances assumed in the first problem given above, and the other with the amount assumed in the second problem; then would the two offices offer the same degree of security, if they set aside as Capital-Reserve the same multiple of the respective average risks. If an office should once decide on thus setting aside a fixed multiple, it could then, as the amount of existing insurances altered in the course of time, change its Capital-Reserve also, according to the above rules; and by retaining the same multiple maintain always the same degree of security. If, for example, an office had reserved for the amount of insurances in Problem (1) ten times the average risk, *i.e.* 11,080 Thalers, then if at another time the amount of insurances became such as is supposed in Problem (2), it would consequently have to reduce its Capital-Reserve to 6180 Thalers. By this means the Capital-Reserve might be safely regulated upon a scientific basis, whereas the usual course of reserving a certain part of the profits, makes the stability of the office depend upon the extent of the profits it may chance to realize— a mode of proceeding so evidently absurd, that we cannot but wonder at its being still followed.

I do not consider it necessary to explain further how the calculation must be made when the lives assured are of different ages, and we have therefore to do with different probabilities of death; for, the theory having been laid down, it must be left to the skill of the actuary to find the shortest way for arriving at a satisfactory approximation in any particular case.

It may perhaps be objected to the preceding method of calculation that it determines the average risk only for a given time, whereas the danger continues until all the insurances have run out. Against this it must be remembered that the amount of the existing insurances will be so essentially altered after a short time by the termination of existing policies on the one hand, and the introduction of new policies on the other hand, that it would be but an idle task to carry on the calculation to the extinction of all the insurances. Such a calculation however for single insurances would be of interest for the purpose of enabling us to compare the individual risks that result from the various kinds of insurances, and thus obtain a rule for loading the net premiums in the different tables of rates. This problem also can easily be solved by the application of the principles above set forth; namely,

by multiplying the possible losses into their respective probabilities, and expressing the value of the average risk by the sum of all the products so obtained.

Before proceeding to a review of the various modes of treatment which our subject has received, it appears necessary to give a summary of the preceding arguments with the view of throwing further light on some important points.

In determining the idea to be attached to the word *Risk* we proceeded on the supposition that there exist in the Table of Mortality no errors—whether such as may have occurred in their compilation, or such as are caused by the unknown deviations from the actual probability of death, that is to say, from the true law of mortality, of which we can only seek for and obtain the most probable values. We have seen that even under this supposition we can never with any certainty expect the mortality to agree with the tables: but can only say that the mortality indicated by the tables is of all others the most probable, provided its occurrence is not physically impossible. On this point we often meet with erroneous views. People too often speak of "accidental deviations" as if the mortality under normal conditions (that is to say, so long as no new causes of death supervene) must of necessity, or with a high degree of probability, conform to the Table of Mortality. We take the statements of the Table somewhat too literally if we say "out of so many born, so many must be still alive after a certain time"; for the Table can only indicate the limit of the proportion remaining alive out of an infinitely large number of persons born. If then among a finite number of persons, no matter how large, another proportion appears, there would yet be not the least ground for perceiving any special causes for it; for though out of many possible events there is one which has a greater probability than any other, yet is it very often more probable that some one or other of the remaining events will happen rather than that particular event. For example, if we have 100 persons of the same age with a probability equal to ·02 of dying within a year, then the probability, that of these 100 persons two will die within the first year, according to the Mortality Table, is only ·27; and we can therefore wager 8 to 3 that that event will not happen.

This single example suffices to show how worthless are all those calculations and conclusions given at such length in the reports of Companies, and exhibiting the deviations from the so-called expected mortality, if these have been in existence only a short time with a small number of insurances in force. A Company may rightly

express regret for the unfavourable mortality it has experienced; but it does wrong to attribute its losses to deviations from the *law*, for that very mortality which *has occurred* is itself the *law*.

It is a very interesting question whether there is in nature such a thing as a fixed Law of Mortality; or in other words, whether, as the number of observations increases, the rates of mortality approximate to a fixed value. The results of statistics, which always relate to fixed numbers and fixed conditions, may indeed establish a strong presumption in favour of the existence of such a Law, but cannot demonstrate it by evidence.

It was however reserved to analysis to answer this question. Poisson has demonstrated that the definite integral which represents the average duration of life, preserves a constant value so long as there is no material alteration in any one of the known or unknown causes of death, including, among others, the various constitutions of new born children, the influences of climate, and the diseases and pecuniary circumstances of the inhabitants of the country. Take for instance an infinite number of various constitutions of new born children, for which the infinitely small probabilities of living exactly the time t are respectively

$$p'dt, p''dt, p'''dt, \ldots.$$

and further let

$$\mathrm{P}', \mathrm{P}'', \mathrm{P}''', \ldots.$$

denote the probabilities of these constitutions. If then, for brevity we put

$$p'\mathrm{P}'+p''\mathrm{P}''+p'''\mathrm{P}'''+ \ldots. = \mathrm{Z}$$

Z will represent a definite Integral, and $\mathrm{Z}dt$ express the probability of t which extends to all possible constitutions.

The mathematical expectation of life of a new born child of unknown physical constitution is thus expressed by the definite integral

$$\int_0^\infty \mathrm{Z}t dt,$$

which has a fixed though unknown value.

If s denote the sum of the ages at which a very great number, n, of individuals born in the same country and at the same time have died, then we have very approximately, and with the highest degree of probability,

$$\frac{s}{n} = \int_0^\infty \mathrm{Z}t dt.$$

This is one of the two remarkable equations which include in

general terms the principle of large numbers, as enunciated by Poisson, who has demonstrated it by the help of a masterly analysis.

The principle of large numbers is the only true point of view from which to contemplate the law of mortality; and a law being adopted, all the probabilities of death are also given, to which we must adhere in all our calculations, so that new hypotheses are neither necessary nor admissible.

From this simple consideration the whole Theory of Risk at once follows, as here introduced, inasmuch as we have adopted the premiums for the insurance against every possible loss as a measure of the danger, and have finally found these premiums to be identical with the average loss or average gain. If we choose to consider the deviations of the mortality from the most probable case as errors, then we have the average error together with the given probabilities of that error, exactly as in observations of natural phenomena, involving measurements, where the probabilities of error are known, and no hypotheses must be made with regard to them as must be done in applying the method of least squares. The average risk is consequently equal to half the average error, because the positive errors or possible gains are not taken into account. This average error is different from the average square of error respecting which Gauss and Bessel have shown that it is a standard wherever the method of least squares can be applied.

The existing works upon Risk are as follows:—

Tetens, Einleitung zur Berechnung der Leibrenten und Anwartschaften (Introduction to the Calculation of Annuities and Reversions). Leipzig, 1786.

Raedell, Lebensfähigkeit von Versicherungs-Anstalten. Berlin, 1857. Pages 219-239.

Bremiker, Das Risico bei Lebensversicherungen (The risk attaching to the grant of insurances on lives). Berlin, 1859.

Zech, Das Risiko bei Lebensversicherungen. Tübingen, 1861.

Lachmund, Das Risiko bei Lebensversicherung (Elsner's Archiv für das Versicherungswesen. 1st Vol. Berlin, 1864. Pages 170-184).

Sprague, On the Limitation of Risks (Journal of the Institute of Actuaries. No. LXIII. April, 1866. Pages 20-39).

J. N. Tetens, Professor of Mathematics and Philosophy at Kiel, was the first to discuss the Theory of Risk. In his above-

mentioned excellent work, which, notwithstanding its antiquated way of dealing with the subject, is still well worth reading, Tetens enters upon a full discussion of the term Risk, and defines its meaning very clearly, without however extending it to the case of any given number of lives insured for different amounts. For the purpose of establishing his theory, the author puts in array a host of mathematical propositions, which my present limits will not allow me to notice. I will only remark that in my opinion modern writers do this deserving man an injustice, when they say that he attaches no fixed idea to the term Risk; for with him the uncertainty is not in his ideas, but in his method. From the greatest possible gain and the greatest possible loss he deduces an approximation to the risk for single annuities; and from these he passes to the risk for a greater number of lives. It will thus be readily understood that his theory must lead to most unsatisfactory results.

The subject of Risk appears to have been neglected for a period of 70 years, when Dr. Raedell brought forward a theory wherein, without giving any reasons for doing so, he transferred to the treatment of Risk certain propositions from the theory of average errors in the method of least squares. Starting from Tetens's definition, Raedell calculates directly for single annuities the average loss that should represent the risk run by the office in the *sale* of an annuity. With respect to the *purchase* of an annuity, he makes use of another method, which is erroneous, and brings out a result differing from the previous one. This difference, the author says, arises from the nature of things. In attempting to estimate the risk for a greater number of insurances, Raedell takes a fatal leap, and simply asserts "that the risk arising out of several contracts is equal to the square root of the sum of the squares of the risks arising out of the single contracts," adding an expression of regret that he was obliged to give this proposition without demonstration, as he had not succeeded in bringing it within the sphere of arithmetical reasoning.

The writers who come after Raedell, with the exception of the Englishman Sprague, likewise employ the method of least squares, treating the subject however very differently.

I propose to give, without further criticism, their principal views; and I will then give some reflections upon the method of least squares, from which it will appear that it is altogether inapplicable to the investigations connected with our subject.

Dr. Bremiker treats in the first place of the risk of single insurances, and identifies it with the square root of the sum of the

squares of all the errors divided by their number, while he forms the squares of the errors themselves by means of the deviations of all the single events from the average or most probable value. This definition therefore differs from Raedell's, but in passing on to the case of several insurances, he follows in the steps of his predecessor. However Dr. Bremiker justly censures the false distinction made by Raedell between purchase and sale.

Dr. Julius Zech, Professor of Mathematics and Astronomy in Tübingen, declares Dr. Bremiker's theory to be erroneous, because the deviations are not accidental but have been already provided for in calculating the premiums, and therefore the method of least squares is not applicable. Zech considers the risk to arise from the circumstance that in reality the rates of mortality and interest are different from those assumed in the calculation. Making certain hypotheses as to the deviations, Zech by means of the method of least squares finds the average error which, in his idea, must represent the risk in the case of single insurances. As to the risk attaching to several insurances, Zech says nothing.

Herr Julius Lachmund opens his treatise with an introduction that promises great things, and closes it with the wish that science may pronounce her verdict upon the construction of his formulæ; as to the principles on which they are based, no man can, in his opinion, entertain any doubt. For the "natural causes" of risk the author looks to the "periodical variations in the rate of mortality"; so that suppositions must be made as to the length of the period in which these fluctuations occur. "Here," says the author, "we may rely upon experience, and in doubtful circumstances, we shall do well not to go too deeply into these suppositions." Further on the author says: "moreover it is not necessary to be too anxious about these suppositions, &c.;" .... "and besides we have it in our power at the end of the next official year to compare the supposition with the actual result, to correct it accordingly, and to draw new conclusions from the basis of this new experience." He then gives some hypotheses, from which the average errors are found and then squared, &c. Then follow new hypotheses as to the proportions of the various periods, on which an experiment already begun by Herr Lachmund is to rest. Then he speaks of the periods in relation to different sums assured, with respect to which he is of opinion that theoretical and withal practically useful propositions will not be discovered without difficulty, while direct investigations will express the law with much greater facility. Then we have more opinions and a couple of examples, followed by the conclusion already mentioned.

Mr. T. B. Sprague, one of the Vice-Presidents of the Institute of Actuaries, gives a variety of propositions about the possible losses and their respective probabilities, without however coming to any conclusion. By "loss" he understands the full payment made by the Company, and demonstrates, among other things, that the total mathematical expectation of loss, as thus understood, is independent of the number of policies, a proposition however which is already well known. The writer however makes no claim to have solved the problem, for he says in conclusion "I have now only to remark that I am well aware that my subject is far from exhausted by the preceding remarks, and that much remains to be done to complete the mathematical part of the inquiry."

Gauss and Legendre, the inventors of the ingenious method of least squares, used it originally for calculating the orbits of planets and comets from observations, which, owing to the imperfection of our senses and the inaccuracy of our instruments, must always be subject to small errors even when made with the greatest precautions and skill. Subsequently its use was extended to all determinations of measure relating to physical science and geodesy. The general problem to be solved by this method, may be briefly enunciated as follows:

From n values of a known function, found by direct observation, for n different sets of values of the variables $u, v, w, \ldots.,$ required to find the system of values of m constants involved in the function, m being less than n, which has, of all possible systems, the greatest probability.

Various analytical processes lead to the conclusion that we must so determine the constants that the sum of the squares of the errors which will occur under the supposition of these values, shall be a minimum. From this it follows that the function which expresses the probability of an error in terms of its magnitude will have the form

$$\frac{h}{\sqrt{\pi}} e^{-h^2x^2} dx,$$

and will express the probability that the error lies within the infinitely small interval x and $x + dx$.

If we consider this function, we are struck with two of its principal properties. In the first place, it evidently presupposes the continuity of the errors; and in the second place, it gives equal probabilities for the positive and the negative errors of the same magnitude. Whence it follows that in the application of this method, if no further correction offers itself, these two con-

ditions must of necessity be fulfilled. Nevertheless we often see it applied to questions that are very far from satisfying these conditions. If I succeed in demonstrating the error of applying it to the determination of the probabilities of death, it will scarcely be necessary to show that it has nothing in common with the Theory of Risk.

I trust I may therefore be permitted to cite a very simple instance, in which two of the most celebrated writers on the subject of Vital Statistics, while agreeing completely as to the admissibility of the method of least squares, yet differ widely in respect of its application, whereas we cannot but think that in this case as with mathematics generally, no difference of opinion can possibly exist if the principles are well established.

In a paper published in Masius's *Rundschau der Versicherungen* (Third year, p. 336) Dr. Heym calculates the probability of death for a given age, from a series of contemporaneous registers of the numbers living and dying, according to the method of least squares. Putting

$$\begin{aligned} n_1 x &= m_1 \\ n_2 x &= m_2 \\ n_3 x &= m_3 \\ \vdots \quad & \quad \vdots \end{aligned}$$

he considers x as the quantity to be determined; $m_1, m_2, m_3, \ldots$ as the observed values of the function, the form of which is known. Consequently he obtains by means of well known rules

$$x = \frac{n_1 m_1 + n_2 m_2 + n_3 m_3 + \ldots\ldots}{n_1^2 + n_2^2 + n_3^2 + \ldots\ldots}$$

Dr. Fischer, in his *Elements of the Science of Life Assurance* (Grundzüge des auf menschliche Sterblichkeit gegründeten Versicherungswesens), Oppenheim a. R. 1860, (pp. 94-98) cannot approve of the fundamental form of equation adopted by Dr. Heym, for which he gives the sufficient reason that *à priori* there are no equal values to determine the amount of the errors whose sum is to be a minimum, inasmuch as they are derived from different numbers of living. Dr. Fischer first tries to determine the relative weight of these observations of different degrees of accuracy, by assuming it to be inversely as the number of the living; but the equations even then do not appear to him equally true, since, other circumstances being the same, the probability of dying will be more correctly derived from large numbers. The way out of this difficulty, the author thinks, is to assume that the weight of the

observations is inversely as the square roots of those numbers. I would now ask, if the whole operation is to depend on the humour of the calculator, to what end have any method at all? Is the method of least squares then not sufficiently established on its true foundation, to guide our steps in safety in its practical application? I should have thought that in the principles of that method as taught by Gauss in his *Theoria motus corporum cœlestium* we were in possession of a firm support to keep us from falling into error in applying it. It is remarkable that all the writers who apply it in connection with rates of mortality or with risk, have not a word to say by way of directing us where we are properly to look for the justification of this application.

We must not conclude however that we shall never, under certain conditions, be able to employ the method of least squares in future investigations into the law of mortality; but we must first have a rigorous demonstration of its applicability, for the mere management of it requires a clear and impartial judgment. We can indeed apply it in accordance with true principles to the example before given, obtaining the same result as we have previously arrived at by the direct investigation of the maximum value of the probability. Suppose we have k observations upon equal numbers of persons living

$$n_1 = n_2 = n_3 \ldots\ldots = n_k = n$$

giving different values of the probability of death, and that we consider the probability of death as the quantity which is subject to errors of observation; then the arithmetical mean of the observed probabilities of death will express the most probable value of the quantity sought, namely,

$$x = \frac{\frac{m_1}{n} + \frac{m_2}{n} + \frac{m_3}{n} + \ldots + \frac{m_k}{n}}{k}.$$

If however, $n_1, n_2, n_3 \ldots n_k$ are of different magnitudes, then the weight of the observations will be proportionate to these quantities, which may be looked upon as representing the number of times the observations are repeated, and we shall therefore have by the help of well known rules:

$$x = \frac{n_1 \cdot \frac{m_1}{n_1} + n_2 \cdot \frac{m_2}{n_2} + \ldots + n_k \cdot \frac{m_k}{n_k}}{n_1 + n_2 + \ldots\ldots + n_k},$$

or simply $$x = \frac{m_1 + m_2 + \ldots\ldots\ldots + m_k}{n_1 + n_2 + \ldots\ldots\ldots + n_k}.$$

The reason why the method must be employed in the above manner, and why it leads to a true result, is not by any means simple, but on the contrary depends upon a subtle analysis that I will content myself with simply indicating. If we ask ourselves the question how great is the probability, that the actual probability of death lies between the limits $\frac{m}{n}+x$ and $\frac{m}{n}-x$, where $\frac{m}{n}$ is the observed probability of death, then in solving it we must express the required probability as a function of m and n. Supposing m and n to increase continually, then will this function approximate to the definite integral

$$\frac{h}{\sqrt{\pi}}\int_{-\delta}^{+\delta} e^{-h^2x^2}dx,$$

where δ denotes the special assumed value of x. We see that this function is exactly the same which, in the method of least squares, expresses the probability that the error lies within two given limits $+\delta$ and $-\delta$; only we must observe that the constant h, which in that method measures the precision, is here put for the expression

$$\sqrt{\frac{n^3}{2m(n-m)}}.$$

What we have said may suffice to show that the subjects we have been speaking of, must be treated with more profound investigations, and considerations of another kind than those hitherto employed. If we wish to carry out a mathematical investigation with success, we must enunciate the problem with precision, treat it in a clear and vigorous manner, and determine at each step the full meaning of our analytical processes. Deceptive analogies and a blind reliance on analytical identities will reduce our reasoning to a dull game of counters, where appearances are taken for reality.

Laplace, in the introduction to his master-work on the Theory of Probabilities, that wonderful monument of mathematical ingenuity, makes the following most appropriate remark: "The theory of probabilities, is virtually nothing but common sense reduced to figures. It enables us to define with precision what clear minds feel by a sort of instinct, often without their being able to explain how, &c."

With the Theory of Risk is connected a question of some practical importance, about which there has been much dispute, without any decided result being arrived at—we mean the well known question: how to calculate the Premium-Reserve for

insurances effected by annual premiums, when the circumstances connected with the management of the business require that the expenses should not be equally distributed over every year of the existence of the policies, but that more expense must be incurred in the first year or at the time of the insurance being effected, than in the following years.

That definition of the reserve which regards it from nearly the same point of view as we have done in the present paper, saying, that "it is formed from the balance of the premiums (improved at compound interest) which at the time of the valuation would not have been expended if the most probable case had constantly happened," leaves no room to doubt that the expenses already incurred, in so far as they are provided for in the premiums, can no more be included in that balance than are the claims which would have been paid under the supposition of the happening of the most probable case. The reserve so determined has this property that, together with the most probable income of the office and accumulation of interest, it exactly provides for the most probable claims and expenses; this being the necessary and sufficient condition of the reserve. In accordance with this principle we must, in making a valuation, on one side of the account set the present value of the future claims, and on the other side the present value of the future gross premiums, less the probable expenses of management. It is clear that the immediate expenses, as for example commuted commissions, inasmuch as they will not again appear among the probable future expenses although included in the gross premiums, may be entered on the credit side as future income, with the same propriety as the net premiums, or those premiums that are to provide for the payment of claims.

The circumstance, that the valuation in this case is not so simple as in the case of a uniform distribution of expenses, where we can use the net premiums on both sides, has given rise to differences of opinion. Some have tried to establish the contrary principle "that the assets and liabilities must both be capitalized by using only the net premiums," a rule which involves the conclusion that the office must consider the expenses incurred at the time of completing the insurances as entirely lost. If, however, we adopt this principle, we evidently disregard the real circumstances of the case, and take up the imaginary point of view of a uniform distribution of expenses; and the principle, if applied to other circumstances, is in contradiction to the true definition of Reserve.

What is the reason then that such views can have been taken up and maintained until now? Our answer is that the idea of Risk has not been strictly defined.

Without a true theory of Risk it was not easy to draw a distinct boundary line between the Premium-Reserve and the Capital-Reserve; the characteristic difference between which consists in the fact that the former forms an integral part of the liabilities; while the latter, as its name denotes, forms part of the capital, and is to serve as a guarantee fund against possible losses in consequence of deviations from the most probable event. It was seen that the payment of a commuted commission, by reducing the amount of the Premium-Reserve in the early years, while the liabilities of the office might become very large, caused a danger; and an endeavour was made to meet this danger by the introduction of a Reserve based on the net premiums, or a so-called Mortality-Reserve, in contradistinction to the necessary Balance-Reserve. This idea of an increased danger is in itself quite right; but, in looking for the means of meeting it, a false conclusion has been come to, from the fact being overlooked that the liabilities of the office will be increased on the one side just as much as the Premium-Reserve on the other, if the Mortality-Reserve is considered necessary for the fulfilment of the engagements of the office. If the office is to be always compelled to have this Reserve in hand, it will thereby be deprived of the means of paying the claims caused by a deviation from the average rate of mortality. But as the Balance-Reserve is sufficient for the average mortality, we have only to form a Capital-Reserve as a protection against possible deviations from the average, which is a question of "Risk," and herein lies the gordian knot to be untied.

Now it has been shown, that the average risk is so connected with the Premium-Reserve, that for each separate insurance the risk diminishes as the Premium-Reserve increases. Should the latter therefore under the given circumstances not increase rapidly enough at the first, the average risk will be so much the greater. The solution of the whole problem thus becomes simple: The greater the expenses in the earlier years, the larger must be the Capital-Reserve.

To look at the question from another point of view, let us suppose that we have a Table of Mortality which agrees with that of the Experience of the 17 English Offices, except that for the 25th year of life it shows a mortality greater by one per cent, and that we wish to calculate by this hypothetical table the uniform annual premium for an assurance of 100 thalers on a life of 24.

It is at once seen that this premium is greater than that found from the Experience Table, and that the present value of the difference is one thaler; because the single premium is higher by one thaler, and this excess has to be equally distributed over the remainder of life. The Mortality-Reserve at the expiration of the first year will evidently not include this thaler, as it is clearly provided for in the future net premiums.

If however this thaler is to pay, not claims, but commission, which is equally provided for in the calculation of the premiums, it is only a change in the kind of payment, which all the insured must bear in common; and this change can have no influence upon the reserve.

The greater expenditure of the first early years, be it for claims, commission, or anything else whatever, cannot affect the principle of the calculation of the Premium-Reserve, but only increase the risk and thus demand a larger Capital-Reserve. If we should look to the Premium-Reserve for a guarantee of the risk, we must of necessity arrive at the conclusion that the profits arising from a favourable mortality must not be divided, in the expectation that in future an unfavourable mortality will compensate for it, while we have already arbitrarily cut the thread that connects the past with the future. In point of fact, there are medical as well as mathematical reasons for this apprehension. If, however, the profits are nevertheless divided, it shows that we do not look to the Premium-Reserve for a guarantee against losses caused by a deviation from the average mortality; and we see withal of what importance to the security of the office a properly adjusted Capital-Reserve must be, for it is not a high Premium-Reserve but a Capital in proportion thereto that insures the stability of an insurance office.

In connection with the subject of commission by single payment, we must further observe that it must be confined within certain limits if the office would not *à priori* expose itself to losses. The principle is simply this: "The Balance-Reserve must never be negative, or what is the same thing, the office must never have to demand any contribution from the assured," for if an assured withdraws either by death or of his own free will while his reserve is negative, then the office loses an amount equal to that negative reserve. To such a loss, however, every office must be exposed which does not confine the payment of commission within the proper limits, however it may calculate its reserve.

German Life Assurance Institute.

IN the number of the *Journal* for April, 1868, Herr Wilhelm Lazarus, of Hamburg, reported the recent establishment in Berlin of a German Life Assurance Institute, a society having generally the same object as the Institute of Actuaries. As the proceedings of such an association cannot fail to interest the members of the Institute of Actuaries and the readers of its *Journal*, we purpose laying before them some of the more interesting papers read before the German Institute; and as a fit introduction to these we now give the following translation of the laws of that Society, for which we are indebted to Mr. J. Hill Williams.—ED. *J. I. A.*

Laws of the Institute for the science of Life Assurance.

§ 1.

Name and Location.

An association shall be established in Berlin under the name *Institute for the cultivation of Life Assurance Science* (Collegium für Lebens-Versicherungs-Wissenchaft).

§ 2.

Object of the Society.

The object of the Institute is the promotion of Life Assurance by the cultivation of those professional and scientific studies which serve as its basis.

§ 3.

Means of attaining that object.

The Institute will endeavour to accomplish this object;

(*a*) By means of the reading of papers upon and the free discussion of, subjects connected with life assurance;

(*b*) By furnishing the fullest possible answers to questions submitted to it appertaining to the science of life assurance;

(*c*) By the establishment of a professional Library;

(*d*) By the publication of the Transactions of the Institute.

(*e*) By all other means which may appear to the Institute calculated to promote the objects of the association.

§ 4.

Members.

The Institute shall consist of Ordinary and Extraordinary Members.

§ 5.

Qualifications of ordinary members.

Any man of good character is eligible as an ordinary member whose writings or business occupations indicate that he is well acquainted, either with the practice or theory of Life Assurance, or of one of the sciences connected with it. A candidate for admission must be proposed by at least two ordinary members; and at the next meeting, the ordinary members shall decide as to his admission by a simple majority of votes.

§ 6.

Duties of ordinary members.

Every ordinary member shall undertake to do his best to promote the objects of the Institute, as stated in § 2; to obey

the laws implicitly, and to pay, in addition to the entrance fee of one thaler, an annual subscription of six thalers, quarterly in advance.

§ 7.

Qualifications of extraordinary members.

Any man of good character may become an extraordinary member, who takes a lively interest in life assurance matters. Admission may be obtained by application to the Committee and the receipt of a card of membership, which is valid for one calendar year, and must be renewed annually. An extraordinary member shall pay two thalers for his card of membership and the same sum for each renewal thereof.

§ 8.

Cessation of membership.

Membership may cease:

(1) By voluntary resignation in writing addressed to the committee;
(2) By expulsion in consequence of the subscription being unpaid for more than half a year after the term when it falls due;
(3) By expulsion on account of objectionable conduct.

In the last case expulsion shall not take place unless at least two thirds of the members present at the Special Meeting having the matter under consideration shall vote for it.

A member shall upon resignation or expulsion lose all his interest in the property of the Institute.

§ 9.

Meetings of the Institute.

The meetings of the Institute shall consist:

(*a*) Of General Meetings (Versammlungen) in which, in accordance with § 3*a* and *b*, papers are read, discussions take place, and answers to questions are given.
(*b*) Of Special Meetings (Sitzungen) in which the special affairs of the Institute are considered, the elections of new members take place, and resolutions are passed.

Ordinary and extraordinary members may attend the General Meetings and ordinary members may introduce visitors.

Ordinary members only are entitled to attend the Special Meetings. In the Special Meetings resolutions shall be carried by a simple majority. Absent members cannot vote by proxy.

Four regular Special Meetings shall be held in the year, on the first Saturday after the beginning of each quarter. Other General and Special Meetings shall be held whenever the Committee think it advisable, or if ten ordinary members send a requisition to that effect to the Committee.

§ 10.

The Committee and its election.

The Committee of the Institute shall consist of five ordinary members, of whom three at least must be resident in Berlin. The election of the Committee shall take place annually in the regular Special Meeting in October, (§ 9) by means of voting-papers, the decision depending upon a simple

majority of votes. In the event of an equality of votes, the question shall be decided by lot, the lots to be drawn by the President. The retiring members shall be re-eligible. The new Committee shall take office on the first of January after their election, and at a Committee-meeting to be held in the beginning of January before the regular Special Meeting, (§ 9), shall nominate, out of the members resident in Berlin, a President, a Secretary and a Treasurer for the current year. The President shall summon all members of the Committee, including those who are not resident in Berlin, to attend the meetings of the Committee, and questions shall be decided by a majority of the members present. If any of the office-bearers are absent, their duties shall be performed by the members present.

§ 11.

Duties of the Committee.

The duties of the Committee shall be to watch over the interests of the Institute at home and abroad; and to carry out the lawful resolutions of the Institute. It shall defray the expenditure required for the objects of the Institute, according to the financial condition to be declared at the beginning of each year, and undertake the management of the Funds. The Committee shall also be responsible for the preparation of the agenda, and the keeping of correct minutes of the proceedings, copies of which shall be duly forwarded to the non-resident members. It shall have charge of the Library and make regulations as to the use of it. The Committee shall also summon and conduct the General and Special Meetings of the Institute. The notices shall be published in the Berlin newspapers and shall state the subjects for discussion.

The Committee shall take care that Papers sent in by ordinary members shall be read with as little delay as possible.

§ 12.

Accounts.

The Accounts and Report for the past year shall be presented, at latest, at the second regular Special Meeting in each year after the first. The accounts shall be audited by a committee consisting of three members to be chosen at the same Special Meeting in which the Committee is elected.

§ 13.

Dissolution of the Institute.

The Dissolution of the Institute shall take place upon the declaration to that effect of at least two thirds of all the ordinary members. In that case the Institute shall dispose of the remainder of its property and of its library either in favour of some scientific Institution or in some other way for purposes of public utility.

§ 14.

Alteration of the laws.

The Laws can only be altered by the resolution of at least two thirds of the members present at a regular Special Meeting. Notices by members relative to any proposed alteration of the laws must be sent in writing to the Committee; and cannot be

disposed of at the next Special Meeting of the Institute unless they shall have been in the hands of the Committee for at least four weeks previous to such Special Meeting.

§ 15. Provisional Committee.

Of the founders of the Society, viz.:

Dr. *Heym* of Leipzig,
Finanzrath *Hopf* of Gotha,
Wilhelm Lazarus of Hamburg,
Dr. *Wiegand* of Halle,
R. Busse, Dr. *Kanner*, Dr. *Zillmer* } of Berlin,

the three last named shall form a provisional Committee until the number of ordinary members residing in Berlin shall exceed ten. In that case, the election of the Committee shall take place, and the Committee so elected shall forthwith undertake the management of the affairs of the Society.

Berlin, 21st December, 1867.

THE SCOTTISH AMICABLE LIFE ASSURANCE SOCIETY.

Established 1826.

SIXTH SEPTENNIAL INVESTIGATION, 1868.

REPORT OF THE ORDINARY DIRECTORS.

The Deaths to be reported for the year 1867 are **159**, and the consequent Claims (including £24,829, 9*s.* 9*d.* of Bonus Additions) amount, after deduction of £12,726, 12*s.* covered by re-assurance, to **£124,606, 12s. 9d.**, which, after deducting the value of the Policies on hand, is much within the sum provided. A large proportion of the Claims has arisen under old Policies, and for two-thirds of the whole amount the Premiums received, with interest at 4 per cent., equal the Claims paid on these, with Additions.

* * * *

" The General Committee of Management order the following Additions to be made " to the Policies participating in the Profits, subject to the provision in the Laws, that no " Addition shall be exigible in respect of any Policy which may emerge within five years " of its date:—

" ' To said Policies an Addition of one and a half per cent. per annum on the amount ' of the Policies, and any additions already made thereto; for the number of ' complete years' standing of the respective Policies, reckoning from the ' termination of previous Additions as to those Policies to which Additions ' were made at last Investigation, and from the dates of the respective Policies ' as to those to which no Additions were made at that time:—

" ' To such of the said Policies as may become Claims by death during the years ' 1868 and 1869, a further Bonus of one and a half per cent. per annum on the ' amount of the Policy, and any Additions thereto, for each additional full year ' of the existence of the Policy.' "

"It is the present view of the General Committee of Management, that during the "remainder of the Septennial period now running, it should, from year to year, "be ordered—if there does not appear reason to the contrary—that all said "Policies becoming Claims before the Investigation as at end of 1874, shall "receive a Bonus at the rate of one and a half per cent. per annum."

After adding the amount of the Bonuses now declared, the Total of the Capital Sums assured by Policies on the Society's books at 31st December last, is **£5,325,772, 10s. 2d.**, the number of Policies 11,328;—the Funds amount to **£1,248,593, 15s. 5d.**, and the Annual Income to **£202,095, 10s. 11d.**

The Members may naturally infer from the documents now submitted that had the same mode of Valuation which was pursued at former Investigations been adopted on the present occasion a larger rate of Addition might have been expected; but the Directors believe that the rate now declared is one of the most satisfactory character, and the alteration, allowed by the Society's Act of Parliament, which has been made on the mode of Valuation, with the sanction of the eminent authorities, Professor De Morgan and Mr. Tucker, will, they expect, ensure its permanent continuance.

MEMORANDUM by MANAGER, *relative to the* VALUATION *of the* POLICIES *of the* SCOTTISH AMICABLE LIFE ASSURANCE SOCIETY *as at 31st December*, 1867.

First,—At last Investigation the Valuation may be stated to have been made principally by the Northampton 4 per cent. Table, but a number of the Classes were valued according to the Premiums of the Society, so as to have in hand what would have been taken from new entrants paying the same Annual Premiums for the same Assurances.

Second,—Upon the present occasion it may be stated, generally, that the Valuation has been made in a way that considerably increases the total value of the Society's Obligations under its Policies.

Third,—A great number of the Classes have still been valued upon the principle, referred to above, of keeping in hand what would have been taken from new entrants paying the same Annual Premiums for the same Assurances. These consist of the following, noted in the Statement of Policies subsisting and their net values:—

Division I.—Participating Policies,—	
All, exclusive of Minimum Premium Classes, except for Single Lives (non-Hazardous), by Annual Premiums, Limited Premiums, and by Single Payments, the values amounting to	£34,795
Minimum Premium Classes, ...	154,589
Division II.—Non-participating Policies,—	
Amount under various Classes, as per Statement .	55,481
Survivorship and Present Annuities, ...	1,530
Endowments and Deferred Annuity, ...	16,111
	£262,506

Fourth,—The Endowment Assurances have been valued in accordance with the Carlisle 4 per cent. pure Table, which, according to the rates

charged by the Society, is equivalent to securing a reservation of 15 per cent. on the net Annual Premiums, the values amounting to £52,660.

Fifth,—The values of the Additions at 31st December, 1860, now existing, amounting to £249,476, 13*s*. 2*d*., have been estimated, according to the Northampton 4 per cent. Table, at £148,261.

Note.—To show the comparative value of such Additions by different Tables, the values of £205,556, 16*s*. 11*d*., being a portion of the above £249,476, 13*s*. 2*d*., applicable to the Class with Additions, by Premiums payable during the whole of life, have been ascertained to amount to the following sums:—

Amount of Additions, £205,556, 16*s*. 11*d*.

Value thereof by	Northampton 4 per cent. Table, . .	£121,444
„	Carlisle 3 per cent. Table, . . .	128,782
„	Carlisle $3\frac{1}{2}$ per cent. Table, . . .	120,441
„	English Life (No. 1. Males,) 4 per cent. Table,	116,251
„	Scottish Amicable Rates, . . .	134,858

Sixth,—The present Annuities have been valued by the Carlisle $3\frac{1}{2}$ per cent. Table, with an Addition to the values of 5 per cent. on account of female lives, the values amounting to £31,694.

Seventh,—The Classes which require principal attention are the remaining—viz., for Single Lives (non-Hazardous), participating in the ordinary way, and the Premiums for which are payable either by Life Annual or Limited Premiums, or have been paid up by Single Premiums, the values amounting to £489,474.

Eighth,—These three Classes, according to the system of last Valuation, as at 31st December, 1860, would have been valued by the Northampton Table. Now they have all been valued by the English Life 4 per cent. pure Table, and the result is to add to the amount which the Valuation would have brought out by the Northampton Table about £26,000.

Ninth,—Much has been said and written about the excellent adaptation for valuation of the Carlisle 3 per cent. "pure" Table, and great labour has been bestowed by the profession in the preparation and publication of tables to assist in working out values according to this basis. The Table was for a long time defended on the ground of its accuracy as regards mortality, and the rate of 3 per cent. interest adopted was at one time defended on the alleged ground that such rate was all that could be fairly calculated upon. Now the view appears to be that it may be admitted 3 per cent. is too low a rate of interest to calculate upon; but that such a rate is required to counterbalance the now alleged too low mortality which the Carlisle Table indicates, for lives, at any rate, which have existed in an office for some time. In the present state of the science, I have proposed, in the case of the three important Classes referred to, to depart from the Northampton 4 per cent. Table, and to adopt one of authority which our Act of Parliament permits us to employ. It is, to say the least, too early to recognize any other Table as so much more suitable as to necessitate an alteration of our rules to obtain its adoption.

Note.—The English Life Table adopted is No. 1. Males. More recent Tables have been issued by the Registrar-General, but the one here adopted is that used by the Society in finding the equivalent single Premium for Annual Premiums (the Annual Premiums being multiplied by the Annuity, according to the English Life Table, No. 1. Males 4 per cent.), and later tables do not, at present, appear to demand any correction of our practice in this respect.

Tenth,—At the same time, although this has added, as stated, to the value about £26,000, I have not recommended any change in the rules by which the present Reserve is fixed.

Eleventh,—Had there been any change to a Valuation by our own rates, or to the Carlisle 3 per cent. "pure," I would have considered it proper to express an opinion that, the Funds on hand being duly estimated, there was no occasion in either case for any Reserve, beyond the so restraining, if necessary, the resulting Bonus, that it might be at such a rate as there would be great probability of maintaining in future.

Twelfth,—So far back as 1844, I intimated my view that a Bonus of 1½ per cent., according to the Septennially accumulating plan of the Society, was as much as could be fairly looked for as a permanent one; and the result of further experience of other Offices of the highest standing again brings prominently to my mind the view that 1½ per cent. should be the standard to be guarded and maintained.

* * * *

Fourteenth,—I may here state that I consider, if the whole Valuation had been made on our own rates, valuing the additions by the Northampton 4 per cent. Table,—

The Total Value of the Liabilities, including Additions, would have been . . .	£1,161,002	13	1
And the Funds being (£1,248,593, 15*s.* 5*d.*, less sums due, £58,703, 16*s.* 7*d.*) .	1,189,889	18	10
There would have been, after giving the additions according to the Northampton 4 per cent. Table, a Surplus of . .	£28,887	5	9

This, I conceive, shows how well grounded is our position.

Fifteenth,—A Carlisle 3 per cent. "pure" Valuation, as it is called, would have produced a somewhat higher liability than our own rates, as regards the principal Class of Policies with Additions, secured by Life Annual Premiums; but I consider our Premiums are entirely sufficient, and in the case of many classes the Valuation by our own rates is considerably more than the value produced by such 3 per cent. Carlisle "pure" Valuation, so that I believe if a full "pure" Valuation had been made by the Carlisle 3 per cent. Table, it would have varied little from that by our own rates.

Sixteenth,—I have *quoted* the word "pure" above, because it is desirable to note that it is too strong an indication of the bearings and results of this mode of Valuation. For instance, it has the undoubted effect, in many cases, owing to the different rate of loading at one age and another, to value an Annual Premium of the same amount payable by the same individual under two Policies at different sums. And it is not clear that the purer way would not be to value the two alike: such would be the result if the rate of loading at the advanced ages were looked to, and the values would be less.

Seventeenth,—Anyhow, while the public cannot readily appreciate the discussions—which however are submitted to them—about "loadings," they can well understand a Valuation on the principle of ascertaining what amount of Single Premiums would be paid by a new body of members similar to that in existence, in addition to equal existing Annual Premiums.

And we have seen (Article Fourteenth) how well the Society can stand the test of this Valuation. Such test is essentially founded on the setting aside Single Premiums according to own rates for the deficiency of the Annual Premiums payable to meet the sums assured at the advanced ages. This mode of Valuation would, no doubt, be unsuitable for many Offices, on account of the unreasonable sums they require for Single Premiums. For instance, the Single Premium on our class without Additions, say at age 40, which regulates the Valuation according to our own rates at that age, is £44, 2*s.* 2*d.*; and the Assurance by payment of this sum is in fact an Assurance of £100, less £44, 2*s.* 2*d.*, by the Annual Interest at 4 per cent. of £44, 2*s.* 2*d.* discounted for one year, or £1, 13*s.* 11*d.* Now, if we take the Annual Premium which the English Life Table referred to would require, at 4 per cent., for Assurance of this sum, we find it is £1, 7*s.* 11*d.* So that we have thus an Addition to the net Premium of 21·4 per cent. But looking to the rates of other Offices, Single Premiums will be found charged as high as 10 per cent. more, producing, on the principle here explained, an Addition to the net Premium of not less than 45 per cent. at 4 per cent interest. What has been said here seems at once to show how safe such a test of Valuation is for us, and at the same time, that the rates charged elsewhere are unreasonable, and prohibitory of a Valuation according to the common-sense principle of "Own Rates."

Eighteenth,—I may here add, that the values of Re-assurances included in the Funds have been estimated in accordance with the Valuation of the Society's Policies; and the Government Life Annuities have been valued by the English Life (No. 1. Males) 4 per cent. Table, which gives a less value than if they had been valued by the Carlisle 3½ per cent. Table, as the Annuities payable by the Society have been.

STATE of the AFFAIRS as at 31*st December,* 1867.

ASSETS,

(*Exclusive of Value of Future Premiums, payable under the Policies.*)

	£	s.	d.
LOANS on Policies (within the extent of their value), ...	£83,814	11	9
" on Heritable (Real) Security, ...	385,031	12	7
" on Debentures, Annuities, heritably secured, and other Property, including Life Rents and Leasehold Property, . . .	65,510	0	7
INVESTMENTS in Terminable Annuities, ...	30,264	8	7
" in Perpetual Annuities, viz., Feu Duties, and Ground Annuals, Guaranteed and Preference Stocks of Railway Companies, and Glasgow Corporation Water Annuities,	478,542	5	0
" in Government Annuities, ...	9,007	13	10
" in Miscellaneous Interests depending on Life, . .	4,851	3	3
" in Property, Glasgow, London, Edinburgh, Dublin, Belfast,	104,350	0	0
AMOUNT INVESTED AT AN AVERAGE INTEREST OF £4, 11*s.* 6*d.* PER CENT.,	£1,161,371	15	7
VALUE OF RE-ASSURANCES, as per Manager's Statement, . . .	41,086	11	6
ANNUAL CONTRIBUTIONS OUTSTANDING at end of the year, less Balance at credit of Agents, ...	25,657	15	9
NOTE.—This amount consists principally of Premiums due in December, the days of grace on which were current at 31st December.			
INTEREST and other Revenue, current from last Payment to 31st December,	11,999	9	3
STAMPS on hand, ...	103	9	6
CASH IN BANK, and Bank Bills on hand, ...	8,374	13	10
Total Assets, ...	**£1,248,593**	**15**	**5**

ENGAGEMENTS.

NET VALUE of Policies for the whole period of Life, ENTITLED TO PARTICIPATE in the Profits, including value of previous Additions, . . .		£827,118 17 7	
NET VALUE of Policies NOT ENTITLED TO PARTICIPATE in the Profits:—			
Assurances, for the whole of Single or Joint Lives, Temporary, Survivorship, and payable at a Certain Age or at Death, .	£108,141 5 2		
Survivorship and Present Annuities, ...	33,224 11 5		
Endowments and Deferred Annuity, ...	16,111 0 11		
		157,476 17 6	
SUM AS PER MANAGER'S VALUATION, .		£984,595 15 1	
FOR SUMS on Policies emerged, but not paid, and other Sums due at 31st December, 1867, . .		58,703 16 7	
Total Engagements, . .			£1,043,299 11 8
Surplus Fund, . . .			**£205,294 3 9**

REPORT by the MANAGER *relative to the Additions which may be made to the Policies.*

It appears from the Report of the State of the Affairs of the Society that the Surplus Fund, as at 31st December, 1867, amounts to	£205,294	3	9
And it appears from the Statement submitted herewith that the value of an Addition, at the rate of £1, 10*s.* per Cent. per Annum on the Sums Assured and previous Additions, is	141,146	15	7
So that if such Addition be declared there would remain	£64,147	8	2

Which, after deducting £3,000, an ample allowance to provide for Prospective Additions for the next two years at the same rate of 1½ per Cent., leaves a Sum exceeding the £50,000 required by the Resolutions of the Society of 17th March, 1857, to be reserved at this Investigation.

OPINION OF PROFESSOR DE MORGAN.

91 ADELAIDE ROAD, LONDON, *May* 18, 1868.

I have paid attention to the state of the Society, as evidenced by the results of the recent Valuation, laid before me by MR. SPENS.

The comparison of this Valuation with the last would have been rendered somewhat difficult, by the proposed substitution of the English Life Table for the Northampton Table (of which the results of calculation are prepared, and have been seen by me), had not the results of the Northampton Table, as applied to this Valuation also, been also laid before

me. This change is no doubt an improvement; the English Table more nearly represents facts than the Northampton Table.

The comparison of the two Valuations, made in the same way, shows to perfect conviction that no sensible alteration has taken place in the state of the Society during the period 1860–67. That is, the effect of the Bonus system, as it has acted during the said period, has been the effect at which every system should aim,—a tendency to equality of benefits; so that £1 assured at a given age, and lasting in the Society through a given period, shall emerge at death with the same Bonus whether the Assurance were made in one year or another.

It is proposed to declare a Bonus of 1½ per cent. per annum for the Septennial period ending with 1867; and a similar Bonus on those Policies which become claims during the present year and its successor. To this plan I give my concurrence precisely on the grounds on which I gave the opinion in favour of it seven years ago. But now, as then, I should object to making a certainty for the term to come. "Who can predict," I asked in May, 1861, "the state of the civilized world for the next seven years?" Since that time we have had the American Revolution and the Cotton Famine.

On the whole conduct of the Valuation I repeat my opinion of May, 1861, adding, as before, decided approbation of the preference now given to the English Life Table.

OPINION OF ROBERT TUCKER, Esq., V.P.I.A.

I have carefully considered the questions submitted to me by Mr. Spens in connection with the Valuation of the Liabilities of your Society on 31st Dec., 1867.

I am of opinion that it will be prudent to introduce the alteration authorized by your Act of Parliament of valuing the classes referred to by Mr. Spens by the English Life Table instead of the Northampton; and

I am also of opinion that the Bonus proposed to be given to the Members, of 1½ per cent. per annum, on the Sums Assured and previous Additions, may be safely declared.

Such a rate of distribution is fully justified by the result of the Valuation, after setting aside the further Reserve required by the Deed of Constitution.

I do not think it would be prudent to declare a higher Bonus than 1½ per cent., on the ground that a Society charging moderate Premiums like the Scottish Amicable, may not be able to maintain a higher rate of distribution; and I am the more disposed to express this opinion, because your Bonus Additions are accumulated septennially, and thus themselves bear Additions (that is, a Bonus on Bonus). Besides, in effect, by the practice of the Society as to prospective or Interim Additions, there is no suspension of Bonus between the time of one investigation and the succeeding one.

I apprehend, too, that fourteen or twenty-one years hence, it is probable the then (generally speaking) less recently selected character of the lives will tend to lessen the Bonus-giving power of the Society, and this is a great reason for keeping the Bonus moderate now, to preserve that uniformity which is so desirable.

I have also been shown the result of a Valuation by the Scottish Amicable Rates. The liability is made to appear considerably greater than by the other Valuation adopted within the limits of the Society's Rules. It is known as the re-insurance method of Valuation, because the sum reserved in respect of each Policy and the Value of its future Premiums are together equal to the single Premium charged by the Society at the present age of the life assured.

This plan is one of the greatest security when carried to the full extent done by MR. SPENS, and it is satisfactory to find that if this Valuation were adopted, the Society would still exhibit a Bonus of 1½ per cent., and a good balance over.

Under every view of the case, therefore, it appears to me that the proposed distribution of 1½ per cent. per annum for the past seven years is the proper Bonus to declare.

70 LOMBARD STREET, LONDON, 18*th May*, 1868.

STATEMENT

SHOWING THE

POLICIES SUBSISTING AND THEIR NET VALUES,

AS AT 31st DECEMBER, 1867.

I.—POLICIES FOR THE WHOLE PERIOD OF LIFE ENTITLED TO PARTICIPATE IN THE PROFITS.	Number of Policies.	Sums Assured.	Annual Premiums.	Annuities.	Net Values.
EXCLUSIVE OF MINIMUM PREMIUM CLASSES.					
Single Lives, by Annual Life Premiums, Limited Premiums, and by Single Payments (Non-Hazardous),	5,925	£2,490,102	£76,692	..	£489,47
,, Classes not included in the above,	362	170,409	7,548	..	26,92
Joint Lives, by Annual Premiums,	123	37,849	1,778	..	7,49
,, other Classes,	2	300	8	..	10
Longest Liver, by Annual Premiums,	4	3,400	41	..	26
Bonus Additions, as at 31st December, 1860,	..	249,477	..	..	148,26
	6,416	£2,951,537	£86,067	..	£672,53
MINIMUM PREMIUM CLASSES.					
Single Lives, by Annual Premiums for Life (Non-Hazardous),	2,631	1,141,408	29,039	..	111,26
,, by Annual Premiums for Life (Hazardous),	160	67,816	2,158	..	6,28
,, by Annual Premiums (Limited),	262	133,280	4,028	..	13,79
,, by Single Payments,	31	32,258	..	..	15,41
,, Classes not included in the above,	126	68,906	2,944	..	6,52
Joint Lives, by Annual Premiums,	21	17,405	611	..	1,10
Longest Liver, by Annual Premiums,	1	2,010	27	..	20
	9,648	£4,414,620	£124,874	..	£827,11

II.—POLICIES NOT ENTITLED TO PARTICIPATE IN THE PROFITS.

	Number of Policies.	Sums Assured.	Annual Premiums.	Annuities.	Net Values.
ASSURANCES FOR THE WHOLE OF SINGLE OR JOINT LIVES, TEMPORARY, SURVIVORSHIP, AND PAYABLE AT A CERTAIN AGE OR AT DEATH.					
Single Lives, ... by Annual Premiums for Life (Non-Hazardous), ..	292	£158,436	£4,774	..	£34,384
" by Annual Premiums for Life (Hazardous), ...	54	53,509	2,737	..	9,356
" by Single Payments, ...	8	6,996	..	..	3,734
" other Classes, ...	16	22,250	629	..	1,052
Joint Lives, ... by Annual Premiums, ...	17	6,261	279	..	1,709
Longest Liver, ... by Annual Premiums, ...	18	15,350	305	..	1,922
Single Lives, Short Period, by Annual Premiums, ...	136	124,390	2,164	..	1,299
" by Single Payments, ...	8	7,245	..	..	162
Survivorship Assurances, by Annual Premiums, ...	21	19,700	381	..	660
" by Single Payments, ...	4	7,418	..	..	426
Miscellaneous, ...	6	9,800	88	..	777
	580	£431,355	£11,357	..	£55,481
Endowment Assurances, by Annual Premiums, ...	753	187,424	8,175	..	52,605
" by Single Payment, ...	1	100	..	..	55
	1,334	£618,879	£19,532	..	£108,141
SURVIVORSHIP AND PRESENT ANNUITIES.					
Survivorship Annuities, Ordinary, by Annual Premiums,	8	..	£280	£763	£1,153
" by Single Payments,	2	..	..	60	317
Do. { Short Period and Education Assurance, } by Annual Premiums,	10	..	26	200	31
by Single Payments,	4	..	..	50	29
	24	..	£306	£1,073	£1,530
Present Annuities, Single Lives	125	..	..	3,827	28,127
" Joint Lives and Survivor, ...	9	..	..	501	3,568
	158	..	£306	£5,401	£33,225
ENDOWMENTS AND DEFERRED ANNUITY.					
Endowments, . . by Annual Premiums, ...	151	£28,655	£1,150	..	£12,993
" by Single Payments, ...	36	3,825	..	..	2,881
Deferred Annuity, by Single Payment, ...	1	..	..	20	237
	188	£32,480	£1,150	£20	£16,111

ABSTRACT.

Net Value of Participating Policies, ...	9,648	£4,414,620	£124,874	..	£827,119
Net Value of Non-Participating Policies:—					
Assurances for the whole of Life, &c., as above, ...	1,334	618,880	19,532	..	108,141
Survivorship and Present Annuities, ...	158	..	306	£5,401	33,225
Endowments and Deferred Annuity, ...	188	32,480	1,151	20	16,111
	11,328	£5,065,980	£145,863	£5,421	£984,596
Additions (proposed to be) Declared as at 31st Dec., 1867,	..	259,793	..	..	141,147
TOTAL SUMS ASSURED, with Additions, ...	11,328	£5,325,773	£145,863	£5,421	£1,125,743

TABLE

SHOWING THE BONUS ADDITIONS ACTUALLY DECLARED ON THE SOCIETY'S POLICIES, TO 31ST DECEMBER, 1867.

Policy opened in Year	£1,000 Policy increased to
	£ s. d.
1827	1,911 10 10
1832	1,780 1 8
1839	1,587 2 6
1846	1,392 5 0
1853	1,240 6 8
1860	1,105 0 0
1866	1,015 0 0

NOTE.—*No Additions are payable on Policies which become Claims within five years of their commencement.*

NOTICES OF NEW BOOKS.

The Law of Fire Insurance. By CHARLES JOHN BUNYON, M.A., of the Inner Temple, Esq., Barrister-at-Law, Author of the "Law of Life Assurance," &c. &c. London: Charles and Edwin Layton, 150, Fleet Street.

We owe an apology to the learned Author of this book for having so long delayed to notice its contents. Except for the appearance of neglect which it carries with it, the delay is not, however, a matter for great regret. We are conscious that nothing is needed from us to commend to those engaged either in the practice of the Law, or the conduct of Fire Insurance business, a work on such a subject by a writer of so well-established a repute.

It cannot be said that the Law of Fire Insurance extends over a very wide area. Mr. Bunyon with all his diligence and research, has found comparatively little to say about it. Omitting such portions of the book as are made up exclusively of Acts of Parliament more or less pertinent, there are 210 pages nominally devoted to its elucidation; but of these, no fewer than 81 are taken up with matters which bear but remotely on the main object of the book, though, perhaps, properly having a place in it. Of this nature, are the obligations to insure against Fire of Landlord and Tenant, Vendor and Purchaser; the "liability for negligence and of the Hundred" for the destruction of property by acts of riot; the statutory and municipal regulations for the prevention and extinguishment of Fire; and the law of Arson. To some extent Fire Insurance Companies have, no doubt, a direct interest in these matters; but, in the main, they are interested only as Life Insurance Societies are interested in the sanitary condition of our towns and in the laws which protect life and limb. Practically, therefore, the remaining 129 pages of the book suffice to

contain a full exposition of the Law of Fire Insurance, and even of some of these, the strict pertinence of their contents vanishes on rigid scrutiny.

We shall not, we trust, be suspected of having said this with any intention of disparaging Mr. Bunyon's labors. We respect them too much to have any such desire. But even legal bricks cannot be made without straw; and that the points are few at which law and the practice of Fire Insurance come into contact, is not a matter on which we can condole either with Mr. Bunyon or the Fire Companies. That these points are really few is, we think, beyond dispute and need occasion no surprise. It follows partly from the position which Fire Offices assume in reference to their business, and partly from the nature of the business itself, that, unlike Life Offices, they enjoy a freedom from the direct action of the Law amounting almost to an immunity.

A Fire Insurance Policy being, probably in its essence but certainly in its form, a contract of indemnity only, "studiously expressed so as to limit the liability, firstly to the value of the property injured or destroyed, and secondly, to the damnification of the Assured," and transferable "to a stranger only in one particular way, namely, by endorsement with the consent of the insurers," a Fire Insurance Company is practically free from the endless entanglements to which a Life Office, *malgré lui*, is constantly subject from the laws of Settlement, of Equitable Assignment, of Insolvency, of bequest, &c. It is true that a Fire Office, settling a claim, cannot be advised to disregard a notice of Assignment "by way of Mortgage," even though it may have been no party to it; but here there frequently comes to its aid the 14 Geo. III., c. 78, s. 83, "which enjoins upon the insurers the expenditure of the Insurance money in reinstating the premises, upon the request of any person interested in the injured buildings"—an Act which the Companies can also themselves set in motion when there exist "grounds of suspicion that the owner or occupier or any other person who shall have insured the same has been guilty of fraud or of wilfully setting fire to the premises." All this tends to free the Companies from those legal trammels which encumber and embarrass Life Offices at every turn. Indeed, it would seem that so far as they are affected by any direct action of the Law outside themselves, they need scarcely ever go into Court. That they so often do is no argument to the contrary. They go chiefly to resist fraud, to which they are subject in common with all the lieges, though doubtless with a special severity. How far they owe this to themselves in the facilities their practice affords for the insurance of fictitious property—of veritable "castles in the air"—we will not stop to enquire, though *they* might do so with very much benefit.

It would appear at first sight that the equitable adjustment of intricate claims and their due apportionment amongst the various offices sharing the risk, was fairly a point on which the peculiar practice of Fire Insurance Companies was necessarily exposed to risk of legal contention; but on consideration it is soon found that arbitration, and not law, ought to afford, as it fortunately does, the proper solution. There are two other points in which the incidence of Fire Insurance differs materially from that of Life Assurance. In the case of a Life Policy, there is never any doubt either as to the nature of the risk or its amount; under a Fire Policy, both are uncertain quantities. We have already seen how a Fire Office may have to go into Court to defend itself against a claim fraudulent on account of

its excessive amount; it has sometimes, also, to do so in order that the Law may determine whether the loss has arisen from a risk contemplated by the Policy. In this latter case there may or may not be an averment of fraud. A fraudulent misrepresentation of the risk, or suppression of material facts, will, of course, vitiate the Policy equally when the Assurance is against Fire as when it is on Life; but there are other misdescriptions and omissions, not tainted by fraud, that may, under the interpretation of legal tribunals, be equally fatal to the Insurance; and loss by Fire may occur under circumstances that were not contemplated by the insurers as part of their risk. In both these forms the nature of the risk may be uncertain, and together they make up the second of the two points of difference to which we have referred. On both these heads, it will be well to hear Mr. Bunyon. At page 59 he thus writes of the necessity of disclosing material facts:

"The principle of *caveat emptor* does not apply to this contract, but "the utmost good faith, *uberrima fides*, must be shown by the assured. "Not only is he required to state all matters within his knowledge which "he believes to be material, but all which in point of fact are so. If he "conceals anything which he knows to be material, it is a fraud; but, "besides, it is said, that if he conceals anything which may influence the "rate of premium which the Insurers may require, although he does not "know that it would have that effect, such concealment entirely vitiates "the policy—the contract being implied that anything material shall be "disclosed. Equity requires that the two parties should contract *pari* "*passu*, which can only be the case when the knowledge of the assured is "communicated. Hence the question whether any fact should be com- "municated depends upon whether it is in itself material, not on the "opinion of the party whether it is so. * * *

"The following well-known case illustrates the doctrine as to the non- "communication of material facts. A fire broke out, upon a Saturday, in "a boat-builder's workshop at Heligoland, and was apparently extinguished "by eight o'clock that evening. It was thought necessary, nevertheless, "to watch the premises, and on the Monday morning it broke out again, "and among other property consumed a warehouse next but one to the "premises which first took fire. On the Saturday evening, when the Fire "was apparently out, after the ordinary mail had started, the owner of "this warehouse sent instructions for its assurance by an extraordinary "conveyance, but failed to communicate the fact of the fire which had "occurred. It was held on general principles, and without reference to "the rules or conditions of the Company, that this concealment rendered "the Policy void."

And on the question of misdescription we find the following, at page 51:

"As will be hereafter particularly seen, it is essential that the property "should be rightly described, and the assured will do well to comply with "the memorandum which is generally indorsed upon the Policy, requesting "him to read his Policy, and if there is any error or misdescription, return "it to the Office for correction. In like manner, he will do well to "examine the conditions of the Policy, and ascertain that they are fully "complied with, and that more particularly in the case of private houses in "the mention in the policy of any stoves or apparatus for warming, which

"the conditions may require to be specially allowed. Lord St. Leonards, "in a note in his Handy Book, gives an amusing description of his "experience on this point. 'A word of advice,' he says, 'about your fire "'insurance. Very few Policies against fire are so framed as to render "'the company legally liable. Generally, the property is inaccurately "'described with reference to the conditions under which you insure. "'* * * * Ascertain that your house falls within the conditions. "'Even having the Surveyor of the Company to look over your house before "'the insurance will not save you, unless your Policy is correct. To "'illustrate this, I will tell you what happened to myself. I have two "'houses in different parts of the country, both of which open by a glass "'door into a conservatory. The one I had insured for a good many "'years, from the time I built it; the other I had insured for a few years, "'from the time I bought it, in the same Office, when a partial fire broke "'out in the latter, and I was then told by the Office (a highly respectable "'one), that my Policy was void, as the opening to the conservatory "'rendered it hazardous, and if so, of course both policies had been void "'from the commencement. I was prepared to try the question, and "'ultimately the objection was withdrawn, and my loss was paid for. "'Upon renewing my Policy with some alterations, I actually had some "'difficulty with the clerk of the company to induce, or rather to force, "'him to add to the description the fact that the drawing rooms opened "'through glass doors into conservatories.' * * * * There can be "no doubt but that his Lordship was a better lawyer than the Manager of "this Company, who contended that the Policy was rendered void by the "omission to mention the conservatory, and that if the point had been "tried, he would have obtained a verdict. But the anecdote well illustrates "the course which should be pursued in effecting an insurance, and "particularly that the assured should take care that the description is "correct, and that the whole contract appears clearly on the face of the "policy, and is not affected by reference to any document which it may "not be in his power to produce hereafter."

The second point—the risk which the Policy is intended to cover—is a large subject, and to it Mr. Bunyon devotes a whole chapter, entitled "The risk and its exceptions." But it is, perhaps, best illustrated in a later chapter, in which, at page 55, the following remarks occur:

"The case of *Pearson* v. *The Commercial Union Insurance Company* "is one of the construction of a specially worded Policy, and may be "considered to prove that the terms of such an instrument cannot be "extended or altered to cover loss, because, by the usage of the insurance "offices, no additional charge would have been made for the privilege of "more extensive terms. The Policy was for £10,000 for three months, "'for the hull of the steamship Indian Empire, with her tackle, furniture, "'and stores on board belonging, lying in the Victoria Docks, London, "'with liberty to go into dry dock, and light the boiler fires once or twice "'during the currency of this Policy.' Adjoining the Victoria Docks "there was a graving dock, not strictly a dry dock, although available as "such, but the entrance was too small to admit the ship; she was therefore "moved two miles up the river to another dry dock, and the lower part of "her paddle-wheels removed to allow of her admission. The repairs being "complete at the end of two months, she was towed down the stream to

"within 500 or 700 yards of the Victoria Docks for the purpose of "having the parts of the paddle-wheels which had been removed replaced "there. The utmost despatch was used, and in 10 days the work was "nearly complete, when she was burnt at her moorings. It was proved "that the premium would have been the same with the principal London "offices, whether the ship lay in the river or in the docks, but that in the "Victoria Docks there were very careful precautions taken against fire— "watchmen at all hours, and a numerous fire brigade, with an ample "supply of water, and all the usual appliances for putting out fires; while "in the river there were only three floating engines, between the arrival "of the first of which and the breaking out of the fire nearly an hour had, "in fact, elapsed. It was also proved that the work might as well have "been done in the dock as in the river, but that the expense would have "been much greater. The Court held that the Policy protected the "vessel while in the Victoria Docks, or any dry dock, whether in the "river or not, and notwithstanding that the latter might be at some "distance from the former, and also while in transition, but that the risk "was limited to the transit, and did not extend to the time during which "the ship stopped in the river not for the purpose of that transit."

We do not think it necessary to enter upon these questions in detail, nor to discuss further the nature and incidence of a Fire Policy, as, except in the particulars named, it is on all fours with a Policy of Life Assurance in its relation to the Law. Nor are we called on to speak at length as to the literary merits of the book. With the exception of an occasional "and which" left by inadvertence to grate on the ear, and the presence of one word, "evil*ly*-disposed," against which we really must enter our protest, there is happily little or nothing to call for criticism. On the other hand, there is all the well-known elegance and perspicuity of style of which Mr. Bunyon is so great a master, lending its charm to an intricate study, which is rendered the less difficult and unengaging by great clearness of arrangement and lucidity of illustration.

CORRESPONDENCE.

ON "TEN YEAR NONFORFEITURE POLICIES."

To the Editor of the Journal of the Institute of Actuaries.

SIR,—Life policies under the above title are described in the July number of the *Journal* (p. 324) as having been largely issued in the United States.

The one point of novelty appertaining to them, is, that "if after two annual premiums have been paid, further payments are to be discontinued, the holder may, upon due surrender of the original policy in accordance with the rules of the Company, receive in lieu thereof a paid-up policy for as many tenth parts of the original sum insured as full annual premiums have been paid."

It may be worth while to investigate the formula for the annual premium necessary to provide for such a risk, and also to examine to

what extent these assurances, in the absence of data as to the probability of surrender, are to be regarded as speculative.

Leaving out of view for a moment the restriction that two annual premiums must be paid before the policy can be surrendered, we will suppose that the surrender can take place at any time. Further, it is obvious that no person assured under this scheme would ask for a paid-up policy at any other time than when a premium became due. Let p_n be the probability at the time the nth renewal becomes payable, that the same will be paid:—then, x being the age at entry, the present value of the liability incurred by the Office at the commencement of each year will be as follows:—

$$\left.\begin{array}{ll}
\text{1st year,} & \dfrac{M_x-M_{x+1}}{D_x} \\[2ex]
\text{2nd} \;\; ,, & p_1\dfrac{M_{x+1}-M_{x+2}}{D_x}+\dfrac{1}{10}(1-p_1)\dfrac{M_{x+1}}{D_x} \\[2ex]
\text{3rd} \;\; ,, & p_1p_2\dfrac{M_{x+2}-M_{x+3}}{D_x}+\dfrac{2}{10}p_1(1-p_2)\dfrac{M_{x+2}}{D_x} \\
\vdots & \qquad\vdots \qquad\qquad\qquad \vdots \\
\text{9th} \;\; ,, & p_1p_2\ldots p_8\dfrac{M_{x+8}-M_{x+9}}{D_x}+\dfrac{8}{10}p_1p_2\ldots p_7(1-p_8)\dfrac{M_{x+8}}{D_x} \\[2ex]
\text{10th} \;\; ,, & p_1p_2\ldots p_9\dfrac{M_{x+9}}{D_x}+\dfrac{9}{10}p_1p_2\ldots p_8(1-p_9)\dfrac{M_{x+9}}{D_x}
\end{array}\right\} \quad . . (1)$$

and if ϖ be the annual premium, the present value of all the premiums payable will be

$$\varpi\left(1+p_1\frac{D_{x+1}}{D_x}+p_1p_2\frac{D_{x+2}}{D_x}\ldots\ldots+p_1p_2\ldots p_9\frac{D_{x+9}}{D_x}\right) \quad . \quad . \quad . \quad (2)$$

The annual premium required will therefore be the sum of the expressions in (1) divided by the coefficient of ϖ in (2).

If in all that precedes we make $p_1=1$, the formulæ will then meet the case where two yearly premiums have to be paid before the privilege of surrender is allowed. In addition to making $p_1=1$ let us suppose $p_2=p_3=p_4\ldots\ldots=p_9$ and denote each of these by p, then (1) becomes

$$\frac{1}{D_x}\Big\{M_x-M_{x+2}+p(M_{x+2}-M_{x+3})+p^2(M_{x+3}-M_{x+4})\ldots+p^7(M_{x+8}-M_{x+9})+p^8M_{x+9}$$
$$+(1-p)\Big(\frac{2}{10}M_{x+2}+\frac{3}{10}pM_{x+3}\ldots\frac{8}{10}p^6M_{x+8}+\frac{9}{10}p^7M_{x+9}\Big)\Big\}$$
$$=\frac{1}{D_x}\left\{M_x-(1-p)\left(\frac{8}{10}M_{x+2}+\frac{7}{10}pM_{x+3}+\frac{6}{10}p^2M_{x+4}\ldots+\frac{1}{10}p^7M_{x+9}\right)\right\} \quad . \; (3)$$

and (2) becomes

$$\frac{\varpi}{D_x}(D_x+D_{x+1}+pD_{x+2}+p^2D_{x+3}\ldots+p^8D_{x+9}) \quad . \quad . \quad . \quad . \quad (4)$$

The annual premium therefore in this case will be found by dividing the coefficient of $\frac{1}{D_x}$ in (3) by the coefficient of $\frac{\varpi}{D_x}$ in (4). The following table exhibits a few numerical values of ϖ, deduced from (3) and (4), corresponding to different assumed values of p. The Carlisle is the table of mortality used, and 3 per cent the rate of interest.

Age at entry.	Annual premium per cent.			
	$p=0$.	$p=\frac{1}{3}$.	$p=\frac{2}{3}$.	$p=1$.
30	4·881	4·867	4·834	4·769
40	5·856	5·848	5·808	5·697
50	6·738	6·749	6·760	6·745

It appears from these results, that when $p_1=1$, and the law of surrender is such that $p_2=p_3=p_4 \ldots =p_9$, the amount of the annual premium is affected very little notwithstanding any change that may be made in the value of p. If the assumed law were known to be true we might conclude with certainty that there is no speculation involved in these assurances but such as might be amply covered by a properly constructed table of premiums. We cannot however tell at what rate surrenders might take place and it will therefore be desirable to examine further into the subject by making some important alteration in the supposed law and comparing the numerical results with those obtained already. Now experience shows that when a surrender takes place it is usually during the earlier years of a policy's existence and but very seldom after it has been in force a lengthened term. In choosing a second hypothesis we will therefore make the following suppositions, namely $p_1=1$, $p_2=p_3=p_4=p_5(=p)$, and $p_6=p_7=p_8=p_9=1$. In this case the present value of the Society's risk is

$$\frac{1}{D_x}\left\{M_x-M_{x+2}+p(M_{x+2}-M_{x+3})+p^2(M_{x+3}-M_{x+4})+p^3(M_{x+4}-M_{x+5})+p^4M_{x+5}\right.$$
$$\left.+(1-p)\left(\frac{2}{10}M_{x+2}+\frac{3}{10}pM_{x+3}+\frac{4}{10}p^2M_{x+4}+\frac{5}{10}p^3M_{x+5}\right)\right\}$$
$$=\frac{1}{D_x}\left\{M_x-(1-p)\left(\frac{8}{10}M_{x+2}+\frac{7}{10}pM_{x+3}+\frac{6}{10}p^2M_{x+4}+\frac{5}{10}p^3M_{x+5}\right)\right\} \quad . \quad . \quad (5)$$

and the present value of the annual premiums is

$$\frac{\varpi}{D_x}\{D_x+D_{x+1}+pD_{x+2}+p^2D_{x+3}+p^3D_{x+4}+p^4(N_{x+4}-N_{x+9})\} \quad . \quad . \quad . \quad (6)$$

The premium, ϖ, is therefore equal to the coefficient of $\frac{1}{D_x}$ in (5) divided by the coefficient of $\frac{\varpi}{D_x}$ in (6).

The following are numerical illustrations of the values of ϖ for various assumed values of p, using the same table of mortality and rate of interest as before.

Age at entry	Annual premium per cent.			
	$p=0.$	$p=\frac{1}{3}.$	$p=\frac{2}{3}.$	$p=1.$
30	4·881	4·865	4·820	4·769
40	5·856	5·843	5·782	5·697
50	6·738	6·748	6·752	6·745

We see from these figures that the variation in the amount of the yearly premium for different values of p is very trifling, and we see moreover, on comparing these figures with those before found that notwithstanding the considerable change made in the suppositions as to surrender, the annual premium is as nearly as possible the same. We may conclude from these results that it is at least highly probable that in such contracts as we have been considering, the speculative element under any circumstances is extremely small.

When $p_1=1$ and $p_2=0$ we get from (1) and (2) $\varpi=\dfrac{M_x-\frac{4}{5}M_{x+2}}{D_x+D_{x+1}}$, and when $p_1=p_2=p_3\ldots=p_9=1$ we find $\varpi=\dfrac{M_x}{N_{x-1}-N_{x+9}}$, the latter being the ordinary formula when a whole life assurance is paid for by ten equal annual premiums.

I am, Sir,
Your obedient servant,
SAMUEL YOUNGER.

316, *Regent Street,*
26th Sept., 1868.

*** We readily give insertion to the above letter on a subject which is not only of theoretical interest but may become of some practical importance. We should have preferred, however, to see the numerical examples worked out by the Experience Table, instead of the Carlisle, when probably some of the irregularities in the results would have disappeared. We should be glad now to see the question treated in another way, viz. by a comparison of the amount of the paid-up policy which the value of the policy would purchase, according to the office single premiums, with the amount of that granted under the regulations quoted above.—Ed. *J. I. A.*

ON A FORMULA IN THE CALCULUS OF FINITE DIFFERENCES.

To the Editor of the Journal of the Institute of Actuaries.

Sir,—I am not about to enter upon the consideration of a theory proposed by some writers, that all mathematical evidence resolves into a perception of identity, and that mathematical propositions are only diversified expressions of the simple formula, $a=a$.* It must however be admitted

* The following is quoted by Dugald Stewart ("Philosophy of the human mind," part 2, cap. 1) from a writer on the subject referred to. *Omnes Mathematicorum propositiones sunt identicæ et repræsentantur hac formulâ*, $a=a$. He adds, "This sentence, which I quote from a dissertation published at Berlin about 50 years ago" (1813), "expresses in a few words what seems to be now the prevailing opinion (more particularly on the Continent) concerning the nature of Mathematical evidence."

that some results which are in the form of equations, approach more nearly to this character than others. In some cases, the equality of two quantities can only be arrived at by means of a long series of intermediate steps, while in others the connexion is more immediate, as in the case of the Binomial theorem $(x+a)^n=x^n+nx^{n-1}a+$&c.; or the more general expression, viz. Stirling's theorem, $fx=fo+f'o.x+f''o.\frac{x^2}{1.2}+$&c. These are not however mere equivalent algebraical expressions, as, for instance, the second side of the latter equation shows the result of the operation on x signified by f on the first side of it. But if we write down such an equation as $a-d=a-b+b-c+c-d$, the character of it will not be altered by making the terms on the second side of it which cancel one another, more numerous or complicated, or even by supposing some law followed in their arrangement.

Now the two fundamental theorems of the Calculus of Finite Differences are

$$u_x=u_0+x\Delta u_0+\frac{x(x-1)}{1.2}\Delta^2 u_0+\text{\&c.}=(1+\Delta)^x u_0 \quad . \quad . \ (1)$$

and $$\Delta^n u_0=u_n-nu_{n-1}+\frac{n(n-1)}{1.2}u_{n-2}-\text{\&c.}=(u-1)^n \quad . \quad . \ (2)$$

If in the first of these equations are substituted the values of $\Delta^n u_0$, from $n=1$ to $n=x$, as deduced from the second equation, the result is the identical equation, $u_x=u_x$.*

Again, according to the property of derivation—one of frequent application in analysis—whereby differences and differential coefficients are treated as the primitive functions of differences or differential coefficients of higher orders, the form of equation (1) will equally hold if for u_x is substituted one of its differences or some function of which u_x is a difference.

Referring now to the demonstration of a formula for interpolation given in vol. xiv., page 244, of this *Journal*, I think the result there arrived at is to be considered not as a fresh property, but rather as involving and illustrating the original properties of the Calculus of Finite Differences. Now when we have arrived at some conclusions, in elementary Geometry for instance, such conclusions are felt to have the same cogency as the axioms and definitions from which they proceed; and in other cases, we

* The distinction I wish to point out between different kinds of equations may perhaps be put in a clearer light as follows. If in the equation, $u_x=(1+\Delta)^x u_0$, have been substituted the values of all the differences previous to $\Delta^x u_0$, as found by means of equation (2) from x terms of the series u_0, u_1, u_2, &c., the former equation is reduced to $u_x=F(u_0, u_1, u_2, \ldots \, u_{x-1}, \Delta^{x-1}u_0)$; and to find $\Delta^x u_0$, the next, namely the $x+1$th, term must be introduced. But this is u_x, the term on the first side of the equation. In other words, the previous terms of the series, and consequently the previous differences, are quite arbitrary in reference to u_x considered only as a *given* quantity, and which might be a term in an infinite number of series. But if $u_x=fx$, a known function of x, it is $=fo+f'o.x+f''o.\frac{x^2}{1.2}+$&c., (Stirling's theorem), and since fo and all the differential coefficients $f'o, f''o, f'''o$, &c., are determined if they are all finite, the value of fx will be found. We must draw a distinction between using a formula containing given differences as an instrument to find unknown terms of a series, and deducing the relation between given terms of a series and their differences as shown in equations (1) and (2); and if the terms of the latter series follow some law, that will furnish a particular case of the general relation alluded to.

should, I think, endeavour, so to speak, to 'account for' results at which we may have arrived. If that be not done, a demonstration may be, to adopt an Aristotelian phrase, one ὅτι, but not one διότι.* I have thought that the result arrived at or rather discovered, I believe, by Mr. Berridge, may be also produced in a different form as follows:

Let S_x denote the sum of x consecutive terms commencing with u_0 of a series u_0, u_1, u_2, &c.; and if we give to x successively the values 0, 1, 2, 3, &c., we have the following:

$$\begin{aligned}
&S_0=0\\
&S_1=u_0\\
&S_2=u_0+u_1\\
&\quad . \quad . \quad . \quad .\\
&S_5=u_0+u_1+u_2+u_3+u_4\\
&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .\\
&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .\\
&S_x=u_0+u_1+u_2+\ \ldots\ldots\ +u_{x-1}
\end{aligned}$$

The first differences of successive terms of the series S_0, S_1, $S_2 \ldots S_x$ thus formed are equal to successive terms of the series u_0, u_1, u_2, $\ldots u_x$; thus $S_1-S_0=u_0$, $S_2-S_1=u_1, \ldots . S_x-S_{x-1}=u_{x-1}$. Let these differences be denoted by δS_0, δS_1, &c., so that the differences of succeeding orders, of S_0, are denoted by δS_0, $\delta^2 S_0$, $\delta^3 S_0$, &c., and let the differences of successive orders, of u_0, in the series u_0, u_1, u_2, &c., be denoted by δu_0, $\delta^2 u_0$, $\delta^3 u_0$, &c., then

$$\delta S_0=u_0,\ \delta^2 S_0=\delta u_0,\ \delta^3 S_0=\delta^2 u_0,\ \delta^4 S_0=\delta^3 u_0 \ldots . \delta^x S_0=\delta^{x-1} u_0 \quad . \quad . \quad (a)$$

Suppose we wish to interpolate intermediate terms of the series S_0, S_1, S_2, &c., between values of S_x taken at successive intervals of p terms from the commencement, viz. between S_0, S_p, $S_{2p} \ldots . S_{mp}$;—for simplicity let $p=5$, and let the number of the orders of differences of the latter terms be 4, and the differences of S_0 of the 1st, 2nd, &c. orders as found from them, be denoted by ΔS_0, $\Delta^2 S_0$, &c. Then the first differences of the series, S_0, S_5, S_{10}, &c., are equal to sums of five consecutive terms of the series, u_0, u_1, u_2, &c., commencing respectively with u_0, u_5, u_{10}, &c., viz.,

$$\begin{aligned}
\Delta S_0&=u_0+u_1+u_2+u_3+u_4\\
\Delta S_5&=u_5+u_6+u_7+u_8+u_9\\
&\text{\&c.}\quad \text{\&c.}\quad \text{\&c.}\quad \text{\&c.}
\end{aligned}$$

But these differences are the same as the quantities denoted by Σ_1, Σ_2,

* To show that this idea of the different modes of proof adopted by mathematical writers is not merely chimerical, I append the following remarks in reference to Dr. Wallis. "Sa façon de démontrer, qui est fondée sur induction plutôt que sur un raisonnement à la mode d'Archimède, fera quelque peine aux novices, qui veulent des syllogismes demonstratifs depuis le commencement jusqu'à la fin. Ce n'est pas que je ne l'approuvre, mais toutes ses propositions pouvant être démontrées *viâ ordinariâ, legitimâ, et Archimedæâ* en beaucoup moins de paroles, que n'en contient son livre, je ne sçai pas pourquoi il a préféré cette manière à l'ancienne, qui est plus convainquante et plus élégante ainsi que j'espère lui faire voir à mon premier loisir." Lettre de M. de Fermat à M. le Chev. Kenelme Digby (Fermat's varia opera Mathematica, p. 191—as quoted by Dugald Stewart, "Philosophy of the human mind," part 2, cap. 9).

&c., in Mr. Berridge's letter, and the other differences $\Delta^2 S_0$, &c., are the same as the differences $\Delta\Sigma_1$, &c., therein, *i. e.*

$$\Delta S_0 = \Sigma_1,\ \Delta^2 S_0 = \Delta\Sigma_1,\ \Delta^3 S_0 = \Delta^2\Sigma_1,\ \Delta^4 S_0 = \Delta^3\Sigma_1 \quad . \quad . \quad . \ (b)$$

It has been shown in vol. xiv., page 23, of the *Journal*,* that

$$\left.\begin{aligned} \delta u_0 &= \frac{1}{5}\Delta u_0 - \frac{2}{5^2}\Delta^2 u_0 + \frac{6}{5^3}\Delta^3 u_0 - \frac{21}{5^4}\Delta^4 u_0 \\ \delta^2 u_0 &= \frac{1}{5^2}\Delta^2 u_0 - \frac{4}{5^3}\Delta^3 u_0 + \frac{16}{5^4}\Delta^4 u_0 \\ \delta^3 u_0 &= \frac{1}{5^3}\Delta^3 u_0 - \frac{6}{5^4}\Delta^4 u_0 \\ \delta^4 u_0 &= \frac{1}{5^4}\Delta^4 u_0 \end{aligned}\right\} \quad (3)$$

Substituting S_0 for u_0 in these formulæ they become

$$\left.\begin{aligned} \delta S_0 &= \frac{1}{5}\Delta S_0 - \frac{2}{5^2}\Delta^2 S_0 + \frac{6}{5^3}\Delta^3 S_0 - \frac{21}{5^4}\Delta^4 S_0 \\ \delta^2 S_0 &= \frac{1}{5^2}\Delta^2 S_0 - \frac{4}{5^3}\Delta^3 S_0 + \frac{16}{5^4}\Delta^4 S_0 \\ \delta^3 S_0 &= \frac{1}{5^3}\Delta^3 S_0 - \frac{6}{5^4}\Delta^4 S_0 \\ \delta^4 S_0 &= \frac{1}{5^4}\Delta^4 S_0 \end{aligned}\right\} \quad (4)$$

Again substituting for the quantities in the latter formulæ, their values as contained in the systems of equations (a) and (b), we have

$$\left.\begin{aligned} u_0 &= \frac{1}{5}\Sigma_1 - \frac{2}{5^2}\Delta\Sigma_1 + \frac{6}{5^3}\Delta^2\Sigma_1 - \frac{21}{5^4}\Delta^3\Sigma_1 \\ \delta u_0 &= \frac{1}{5^2}\Delta\Sigma_1 - \frac{4}{5^3}\Delta^2\Sigma_1 + \frac{16}{5^4}\Delta^3\Sigma_1 \\ \delta^2 u_0 &= \frac{1}{5^3}\Delta^2\Sigma_1 - \frac{6}{5^4}\Delta^3\Sigma_1 \\ \delta^3 u_0 &= \frac{1}{5^4}\Delta^3\Sigma_1 \end{aligned}\right\} \quad (5)$$

The coefficients in (3) and (5) are the same; but the two systems of equations cannot coexist, because in (4) $\delta^5 S_0 = \delta^4 u_0 = 0$, and this is an equation of condition for (5).

I am, Sir,
Yours obediently,
THOMAS CARR.

7, *Royal Exchange*,
6*th November*, 1868.

* The same results are also given by Dr. Farr, vol. ix., page 136, but in a different form.

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C. & E. Layton, Printers, 150, Fleet Street; and 12, Gough Square.